学位申請論文

On quartic surfaces and sextic curves with singularities of type $\tilde{E}_8$, $T_{2,3,7}$, E_{12}

卜部　東介

学 位 審 査 報 告

氏　　　　名	ト　部　東　介
学 位 の 種 類	理　学　博　士
学 位 記 番 号	論 理 博 第　　　号
学位授与の日付	昭和　　年　　月　　日
学位授与の要件	学位規則 第5条 第2項該当

（学位論文題目）

On quartic surfaces and sextic curves with singularities of type $\tilde{E}_8$, $T_{2,3,7}$, E_{12}

（特異点 $\tilde{E}_8$, $T_{2,3,7}$, E_{12} を持つ四次曲面、六次曲線について）

論文調査委員	主 査　広 中 平 祐 中 野 茂 男　島 田 信 夫

理　学　研　究　科

氏名	ト　部　東　介

（論文内容の要旨）

3次元射影空間 $\mathbb{P}^3$ の中の二次曲面は容易に分類でき、その特異点も単純である。$\mathbb{P}^3$ の中の三次曲面の特異点、特に孤立した特異点の分類とその配置は1934年にDv Valによって解決されたが四次曲面となると特異点の分類と配置は急激に複雑さを増す。申請者の主論文はこの問題に関して画期的な成果を得た。Dv Val の特異点は、1966年にM. Artin によって有理型2重点という概念のもとに抽象化され一般化されて完全な分類を得た。その後、有理型特異点の理論は、Brieskorn、斉藤恭司、Lanfer，Yau，Arnold，成木，申請者など多数の研究者によって様々な代数幾何学的視点から研究が進められ、有理型でないより複雑な特異点の分類理論に進展した。同時に、それ等特異点の変形理論、Moduli 理論、Lie 群や Coxter 群との関係等も、上記研究者の他Pinkham，Looijenga，Merindal の成果を得て、次第に解明されてきた。

申請者は、これらの成果を踏台として独自の代数幾何学的手法を加え、$\mathbb{P}^3$ の中の四次曲面の孤立した特異点の分類と配置、Torelli 型のModuli 理論混合Hodge 理論などを詳細にわたって研究し、明解な成果を得ている。特に四次曲面が $\tilde{E}_8$，$T_{2,3,7}$，E_{12}のいづれかの型の特異点を少くとも一つもっている場合、他の特異点の配置に明解な記述方法を確立している。例えば、$\tilde{E}_8$ をもつ場合、その他の特異点の配置はもう一つの $\tilde{E}_8$ だけか、または、Dynkin グラフ B_9 または E_8 から出発して特定の初等的操作を繰返すことによって出来る有理型2重点の配置である。$T_{2,3,7}$，E_{12} の場合も、同様な特異点配置の可能性に一覧表を得ることに成功した。申請者は、三次曲面の場合から類推することは至難といえる四次曲面の場合の特異点配置に関して、主論文で幾多の難題を見事に解決しており、その成果は重要である。

氏名	ト部東介

（論文審査の結果の要旨）

1970年代に入って以後、曲面上の特異点の研究は、多数の研究者による活発な研究によって代数幾何学の分野では最も著しい発展を遂げているものの一つである。その中で、主論文や参考論文で発展された結果で明らかなように、申請者の研究はユニークであり、その成果は重要である。特に主論文では、四次曲面の特異点配置のみならず、それと密接な関係のある六次曲線の特異点配置を解明しており、さらにそれ等に関する定理の証明を用いた手法は、四次曲面や六次曲線のModuliの層構造を解明する上で有効であることが示されている。特異点をもった四次曲面や、$\mathbb{P}^2$の2次Coveringで六次曲線を分岐曲線とする曲面の研究は、代数幾何学で著名な研究課題であるK3曲面のTorelli型Moduli空間の問題と密接な関係をもつものである。その意味でも、申請者の研究成果は、代数幾何学の重要課題に貢献するところ大である。主論文を中心とした特異点配置の研究のみならず、それ以前にも代数幾何学の他の問題に重要な結果を得ており、申請者の秀れた研究能力は充分に示されている。

以上を総合して、本論文は理学博士の学位論文として価値あるものと認める。

なお、主論文及び参考論文に報告されている研究業績を中心とし、これに関連した研究分野について試問した結果、合格と認めた。

学位申請論文の要約

３次元射影空間内の孤立特異点のみを持つ４次曲面、あるいは２次元射影空間内の重複成分を持たない６次曲線を考える。ここでは特にそれが単純楕円型特異点$\tilde{E}_8$、カスプ特異点$T_{2,3,7}$、ユニモジュラー例外型特異点E_{12}を持つ場合を考える。この論文では、ほかにどのような特異点が現れるかが、ディンキン図形で記述される法則に従っていることを示す。

この論文ではすべての多様体は複素数体$\mathbb{C}$上定義されていると仮定する。

定義　１　　いくつかの連結なディンキン図形からなる集合が与えられたとしよう。次の手続をその初等変換という。

（１）それぞれの成分を拡大ディンキン図形でおきかえる。

（２）そのあとで、それぞれの成分から任意にえらんだひとつ以上の頂点とそれを結ぶ辺を消し去る。

二重線を含まぬディンキン図形（Ａ、Ｄ、Ｅタイプ）は曲面上の有理二重点に対応することに注意しよう。

定理　２　　３次元射影空間内の４次曲面Ｘが孤立特異点のみを持つと仮定する。さらに単純楕円型特異点$\tilde{E}_8$をもつと仮定する。このときＸ上の特異点の種類と数は$\tilde{E}_8$にたすことのつぎのもののうちのひとつである。

（Ｉ）ディンキン図形B_9から初等変換２回でえられた図形の集合で短ルートに対応する頂点を残していないものに対応するもの。

（ＩＩ）ディンキン図形E_8から初等変換２回でえられた図形の集合に対応するもの。

（ＩＩＩ）もうひとつ$\tilde{E}_8$。

逆に上の（Ｉ）、（ＩＩ）、（ＩＩＩ）に現われたもの、たすことの$\tilde{E}_8$は必ず３次元射影空間内の孤立特異点のみを持つ４次曲面上に実現できる。

注意　滑らかな曲面上の滑らかな楕円曲線で自己交点数が－１のものを一点につぶしてえられる特異点が$\tilde{E}_8$である。

定理　３　　（および定理　４．　かっこ内の記述で定理　４をあらわす。）　３次元射影空間内の孤立特異点のみを持つ４次曲面を考える。さらにカスプ特異点$T_{2,3,7}$（ユニモジュラー例外型特異点E_{12}）をもつと仮定する。このときＸ上の特異点の種類と数は$T_{2,3,7}$（E_{12}）にたすことのつぎのもののうちのひとつである。

（Ｉ）ディンキン図形D_9（A_8）の部分図形に対応するもの。

（II）拡大ディンキン図形$\tilde{E}_8$の真部分図形（ディンキン図形E_8の部分図形）に対応するもの。

逆に上の（I）、（II）に現われたもの、たすことの$T_{2,3,7}$（E_{12}）は必ず3次元射影空間内の孤立特異点のみを持つ4次曲面上に実現できる。

注意． 1 まったく違う2つのものが同じ名前$\tilde{E}_8$で呼ばれていることに注意しよう。ひとつは特異点であり、ひとつは拡大ディンキン図形である。

2． 滑らかな曲面上の有理曲線で通常二重点（通常カスプ）をただひとつもち、自己交点数が－1のものを一点につぶしてえられる特異点が$T_{2,3,7}$（E_{12}）である。

3 上の（I）の内容は次のように言ってもよい。「（I）ディンキン図形B_9から初等変換<u>1回</u>でえられた図形の集合で短ルートに対応する頂点を残していないものに対応するもの。（ディンキン図形B_9の部分図形で短ルートに対応する頂点を含んでいないものに対応するもの。）」

4． 上の（II）も、もちろん初等変換という言葉を使って言い直すことができる。

5 初等変換という言葉を使って言い直した時のディンキン図形の拡大の回数、2、1、0は、実はそれぞれ特異点$\tilde{E}_8$、$T_{2,3,7}$、E_{12}の極小特異点除去における例外集合の基本群π_1のランクに対応している。

さて方程式$f(x, y)=0$で原点に定義される平面曲線の特異点を式$z^2-f(x, y)=0$で原点に定義される曲面の特異点と同じ名前でよぶことにしよう。もし、$z^2-f(x, y)$と$z^2-g(x, y)$が座標変換で移りあうのなら、$f(x, y)$と$g(x, y)$も座標変換で移りあうことが証明できるから、そうしてもさしつかえないことがわかる。

定理 5 （i） 2次元射影空間内の重複成分を持たない6次曲線Bを考える。さらに単純楕円型特異点$\tilde{E}_8$をもつと仮定する。このときB上の特異点の種類と数は$\tilde{E}_8$にたすことのつぎのもののうちのひとつである。

（A）ふたつの成分を持つディンキン図形E_8+A_1から初等変換2回でえられた図形の集合に対応するもの。

（B）もうひとつ$\tilde{E}_8$であるか、もうひとつ$\tilde{E}_8$たすことのA_1。

逆に上の（A）、（B）に現われたもの、たすことの$\tilde{E}_8$は必ず2次元射影空間内の重複成分を持たない6次曲線上に実現できる。

（ii） 次の10とおりの特異点の種類と数を持つ2次元射影空間内の重複成分を持たない6次曲線の集合は、すべての平面6次曲線の集合$\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$のなかで非連結である。

<1> $\tilde{E}_8+A_7$ <2> $\tilde{E}_8+2A_3$

<3> $E_8+A_5+A_1$ <4> $E_8+A_3+2A_1$
<5> E_8+4A_1
<6> $E_8+A_7+A_1$ <7> $E_8+2A_3+A_1$
<8> $E_8+A_5+2A_1$ <9> $E_8+A_3+3A_1$
<10> E_8+5A_1

定理 6．（および定理 7 かっこ内の記述で定理 7をあらわす。） 2次元射影空間内の重複成分を持たない6次曲線Bを考える。さらにカスプ特異点$T_{2,3,7}$（ユニモジュラー例外型特異点E_{12}）をもつと仮定する。このときB上の特異点の種類と数は$T_{2,3,7}$（E_{12}）にたすことのつぎのもののうちのひとつである。
（A）ふたつの成分を持つディンキン図形E_8+A_1から初等変換１回でえられた図形の集合に対応するもの。
（ディンキン図形E_8の部分図形に対応するもの。）
逆の（A）に上に現われたもの、たすことの$T_{2,3,7}$（E_{12}）は必ず2次元射影空間内の重複成分を持たない6次曲線上に実現できる。

以上が主要結果であるが、その証明は非常に複雑である。その中で次の点が本質的であると思われる。
1　6次曲線の場合はそれにそって分岐する二重被覆を考えて、曲面論にもちこむ。
2　E_8、$T_{2,3,7}$、E_{12}をひとつもつ場合は考えている曲面Xの極小特異点除去Zは有理曲面であり、2次元射影空間から１０回のブロー・アップでえられることがわかる。
3　Z上に定まる有理2－形式にたいして、K3曲面の周期の理論のまねをすると、Zのモジュライ空間が具体的に構成できる。
4　Zの2次コホモロジー群がルート格子を含んでいる関係でモジュライ空間にワイル群がはたらく。そして、問題はその作用の固定点を調べることに帰着する。
5．　4次曲面の場合、モジュライ空間の点が4次曲面として実現できないで、2次曲面の上の分岐二重被覆になってしまったり、埋め込みを決めるリニアー・システムが固定点をもってしまったりすることが、ちょうど、短ルートに対応する鏡影変換の固定点となることに対応することがわかる。
6．　さらにユークリッド空間へのワイル群の作用で基本領域をもつものの解析に還元される。
7　不等式つきのディオファントス方程式を解かねばならぬところがあるのだが、１０回のブロー・アップの場合には具体的に解くことができる。また、第一種例外曲線の処理が１０回のブロー・アップの場合には非常にうまくいく。

我々の定理は４次曲面、６次曲線のうち特殊なものしか扱っていない。もちろん、有理特異点ばかりを持つ場合や、他の悪い特異点を持つ場合にも同様の定理があると予想するのであるが、いまのところわからない。今後少しずつ、扱える範囲を拡大していきたいと考えている。

On quartic surfaces and sextic curves with singularities of type $\tilde{E}_8$, $T_{2,3,7}$, E_{12}

By Tohsuke Urabe

Tokyo Metropolitan University

Department of Mathematics

Fukazawa, Setagaya, Tokyo, 158, Japan

§ 0. Introduction.

In this article we take up normal quartic surfaces in $\mathbb{P}^3$ and reduced sextic curves in $\mathbb{P}^2$. Especially we would like to treat the case where they have a simple elliptic singularity $\tilde{E}_8$, a cusp singularity $T_{2,3,7}$, or a unimodular exceptional singularity E_{12}. (Cf. Arnold [1], Saito [18]) We show that when they have such a singularity and other several singularities, the configuration of singularities is subject to a certain law explained from the viewpoint of Dynkin graphs. Indeed we will verify the following theorems. Now in this article we assume that every variety is defined over the complex number field $\mathbb{C}$.

Definition 0.1. For a given set of several connected Dynkin graphs, the following procedure is called an <u>elementary transformation</u> of it.

(1) We replace each component by the extended Dynkin graph of the corresponding type.

(2) After that, we take away arbitrarily chosen one or more vertices and their connecting edges from each component.
(Cf. Bourbaki [3], Dynkin [6])

Note that any Dynkin graph without multiple lines is associated to a rational double point on a surface. (Cf. Artin [2])

Theorem 0.2. Assume that a normal quartic surface X (i.e. a surface of degree 4 with only isolated singular points) in the projective space $\mathbb{P}^3$ of dimension 3 has a simple elliptic singularity $\tilde{E}_8$. Then the configuration of singularities on X is $\tilde{E}_8$ plus one of the following.

(I) a configuration of rational double points associated to a set of Dynkin graphs which is obtained from the Dynkin graph B_9 by elementary transformations repeated <u>twice</u> in such a way that the resulting set of Dynkin graphs has no vertex corresponding to a <u>short root</u>.

(II) a configuration on rational double points associated to a set of Dynkin graphs obtained from the Dynkin graph E_8 by elementary transformations repeated <u>twice</u>.

(III) another $\tilde{E}_8$.

Conversely every configuration appearing in the above (I), (II), (III) plus $\tilde{E}_8$ can be realized on a normal quartic surface in $\mathbb{P}^3$ as singularities.

Remark. 1. The singularity obtained by contracting a smooth elliptic curve with the self-intersection number -1 on a smooth surface is the singularity $\tilde{E}_8$.

2. In case (III) two elliptic curves appearing on the resolution of singularities on X are isomorphic. This is Y. Umezu's result. (Cf. Umezu [21])

3. Note that the elementary transformation defined by Dynkin in [7] and our elementary transformation is slightly different.

4. In particular consider the case where after the first elementary transformation the unique vertex θ in B_9 corresponding to the short root is left and however the connecting multiple edge is eraced. In this case in the first stage of the second elementary transformation, θ is replaced by the extended graph $\tilde{A}_1$. Then note that as an agreement we regard both vertices of $\tilde{A}_1$ as ones corresponding to short roots.

Theorem 0.3. (resp. Theorem 0.4.) Consider a normal quartic surface in $\mathbb{P}^3$ with a cusp singularity $T_{2,3,7}$. (resp. an exceptional singularity E_{12}) The configuration of singularities on X is $T_{2,3,7}$ (resp. E_{12}) plus one of the following.

(I) a configuration of rational double points associated to a subgraph of the Dynkin graph D_9. (resp. a subgraph of the Dynkin graph A_8.)

(II) a configuration of rational double points associated to a proper subgraph of the extended Dynkin graph $\tilde{E}_8$. (resp. a subgraph of the Dynkin graph E_8.)

Conversely every configuration in the above (I), (II) plus $T_{2,3,7}$ (resp. E_{12}) can be realized on a normal quartic surface in $\mathbb{P}^3$ as singularities.

Remark. 1. Note that two different objects are called by the same name $\tilde{E}_8$. One is a surface singularity and the other is the extended Dynkin graph.

2. The singularity obtained by contracting an irreducible rational curve with an ordinary double point (resp. an ordinary cusp) with the self-intersection number -1 is $T_{2,3,7}$. (resp. E_{12})

3. (I) is equivalent to saying that ' a set of graphs with no vertex corresponding to a short root obtained from the Dynkin graph B_9 by one elementary transformation'. (resp. ' a subgraph of the Dynkin graph B_9 with no vertex corresponding to a short root') In section 5 we see that the Dynkin graph B_9 is the essential one.

4. Of course we can restate (II) using the word 'elementary transformation', too.

5. We will see that the number of extensions 2, 1, 0 in Theorem 0.2, Theorem 0.3, Theorem 0.4 respectively is the rank of the fundamental group π_1 of the exceptional curve in the minimal resolution of the singularity $\tilde{E}_8$, $T_{2,3,7}$, E_{12} respectively.

Now we call a plane curve singularity defined by $f(x, y) = 0$ at the origin by the same name as the surface singularity defined by

$z^2-f(x, y) = 0$ at the origin. (Thus there is a rational double point which is by no means a double point as a curve singularity. —— D_ℓ, E_6, E_7, E_8——. Moreover it is known that the right-equivalence class of $f(x, y) = 0$ is uniquely determined by that of $z^2-f(x, y) = 0$.)

Theorem 0.5. (i) Let B be a reduced sextic curve in the projective space $\mathbb{P}^2$ of dimension 2. (i.e. a plane curve of degree 6 without multiple components) Assume that B has a simple elliptic singularity $\tilde{E}_8$. Then the configuration of singularities on B is $\tilde{E}_8$ plus one of the following.

(A) a configuration of rational double points associated to a set of Dynkin graphs obtained from the Dynkin graph E_8+A_1 by elementary transformations repeated twice.

(B) either another $\tilde{E}_8$ or anther $\tilde{E}_8$ plus one A_1.

Conversely every configuration appearing in the above (A), (B) plus $\tilde{E}_8$ can be realized on a reduced sextic curves as singularities.

(ii) The set of reduced curves with any one of the following configuration of singuralities has two or more connected components in the space of all sextic curves $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$.

<1> $\tilde{E}_8+A_7$ <2> $\tilde{E}_8+2A_3$ <3> $\tilde{E}_8+A_5+A_1$ <4> $\tilde{E}_8+A_3+2A_1$
<5> $\tilde{E}_8+4A_1$ <6> $\tilde{E}_8+A_7+A_1$ <7> $\tilde{E}_8+2A_3+A_1$ <8> $\tilde{E}_8+A_5+2A_1$
<9> $\tilde{E}_8+A_3+3A_1$ <10> $\tilde{E}_8+5A_1$

Theorem 0.6. (resp. Theorem 0.7.) Consider a reduced sextic plane curve B with a cusp singularity $T_{2,3,7}$. (resp. a unimodular exceptional singularity E_{12}.) Then the configuration of singularities on B is $T_{2,3,7}$ (resp. E_{12}) plus a configuration of rational double points associated to a proper subgraph of $\tilde{E}_8+A_1$ which is not equal to $\tilde{E}_8$. (resp. a subgraph of the Dynkin graph E_8.)

Conversely such configurations are realized on reduced sextic curves.

The study of projective varieties and their singularities has long history and it has been done from various view-points. From among them let us pick up some results deeply connected with this article. In 1934 Du Val found out that configuration of singularities on cubic surfaces, plane quartic curves and sextic curves on a singular quadric surface in $\mathbb{P}^3$ can be classified from the view-point of so-called Coxeter groups and root systems of E-type. (Du Val [22]) His result was rediscovered by modern mathematicians from a different point of view during 1970's. (Pinkham [16], Looijenga [10], Mérindol [13], Naruki, Urabe [15]) In particular taking up related topics Looijenga established a Torelli-type theorem for rational surfaces with effective anti-canonical divisors by the mixed Hodge theory and integration of rational 2-forms. His theorem is a powerful tool to study them. (Looijenga [10]) On the other hand Shah classified singularities on quartic surfaces from the view-point of the geometric invariant theory. (Shah [20]) An example of non-ambient-isotopic sextic curves was given in Zariski [24].

The results in this article will be mainly obtained by developing the above-mentioned Looijenga's method further.

The contents of this article is like the following. Section 1 is the preliminary part. We explain that the study of sextic curves B is reduced to the study of branched double covering X of $\mathbb{P}^2$ branching along B and that such branched coverings and quartic surfaces with anti-canonical divisors and ruled surfaces with positive irregularity. From section 2 to section 5 we study rational surfaces. In section 2 we explain a generalized version of Looijenga's Torelli-type theorem. Our version does not use integration of 2-forms explicitly and it is easier to understand, we think. As a result we have an algebraic group $\mathrm{Hom}(\Gamma, E)$ as a moduli space of a certain class of rational surfaces, where Γ is a certain free $\mathbb{Z}$-module with a bilinear form and E is either an elliptic curve with a group law, a multiplicative group $\mathbb{C}^*$, or an additive group $\mathbb{C}$. In addition the relation between our version, theory of integration and the mixed Hodge theory is explained. Section 3 is devoted to study properties of linear systems on them. Section 4 is the Diophantine theoretic part. We determine the class of the polarization in the Picard group. The action of the Weyl group on $\mathrm{Hom}(\Gamma, E)$ is studied in section 5. The case of ruled surfaces with positive irregilarity is taken up in section 6.

I would like to express my heartily thanks to my teachers and colleagues. In particular we thank Mr. T. Fukui for pointing out an error in the first version of this article.

Now we guess that our theorem is a small part of a big theorem dominating all quartic surfaces and all sextic curves, of course. There are two reasons we take up only surfaces with $\tilde{E}_8$, $T_{2,3,7}$, E_{12} here. One is that since most of them are rational, they have a rather simple global structure. The other is that the fundamental domain of the Coxeter group introduced in section 2 is easier to handle than that in other cases. Therefore the next problem should be the next step of our study.

Problem. Find out the general law explaining which singularities appear on quartic surfaces and sextic curves.

For line bundles L, M and divisors A, B on a smooth surface Z, the intersection number is denoted by $L \cdot M$, $L \cdot A$, or $A \cdot B$ in this article. Sometimes we write L^2, A^2 instead of $L \cdot L$, $A \cdot A$. The complete linear system associated to the line bundle L is denoted by $|L|$. The complete linear system $|\mathcal{O}_Z(A)|$ associated to a divisor A is denoted by $|A|$ for brevity. If M is a dual line bundle of L, we denote $|M|$ by $|-L|$.

§ 1. Preliminaries.

In this section we explain that quartic surfaces and branched double coverings of $\mathbb{P}^2$ branching along sextic curves are roughly classified into 3 types; K3 surfaces, rational surfaces and ruled surfaces with positive irregularity.

Let X be a quartic surface (i.e. a surface of degree 4) in a 3-dimensional projective space $\mathbb{P}^3$ with the structure sheaf $\mathcal{O}_X$. We assume that X is normal. Normality is equivalent to that X has only isolated singularities in this case. (Cf. Matsumura [12]) Every local ring of X is not only Cohen-Macaulay but also Gorenstein. Thus we can define the dualizing invertible sheaf ω_X on X. (Cf. Hartshorne [8])

Lemma 1.1. For a quartic surface X, we have

(1) ω_X is a trivial invertible sheaf, i.e., $\omega_X \cong \mathcal{O}_X$.

(2) $H^1(\mathcal{O}_X) = 0$.

Proof. (1) $\omega_X \cong N_{X/\mathbb{P}^3} \otimes \omega_{\mathbb{P}^3}|_X \cong (\mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^3}(-4))|_X \cong \mathcal{O}_X$, where $N_{X/\mathbb{P}^3}$ is the normal bundle of X.

(2) It follows easily from the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

since $H^1(\mathcal{O}_{\mathbb{P}^3}) \cong H^2(\mathcal{O}_{\mathbb{P}^3}(-4)) \cong 0$. Q.E.D.

Let $\rho: Z \longrightarrow X$ be the minimal resolution of singularities of X. We have the Leray spectral sequence

$$E_2^{p,q} = H^p(R^q\rho_*\mathcal{O}_Z) \Longrightarrow H^{p+q}(\mathcal{O}_Z)$$

Note that the support of $R^1\rho_*\mathcal{O}_Z$ is contained in the set of singular points of X. The geometric genus of a singilar point $x \in X$ is defined by $p_g(X,x) = \dim_{\mathbb{C}}(R\rho_*\mathcal{O}_Z)_x$. It is known that $p_g(X,x)$ is well-defined. (Wagreich [23]) Moreover $p_g(X,x) = 0$ if and only if $x \in X$ is either a smooth point or a rational double poit. (Artin [2])

Lemma 1.2. $$\chi(\mathcal{O}_Z) + \sum_{x \in X:\text{singular points}} p_g(X,x) = \chi(\mathcal{O}_X) = 2$$

where $\chi(F)$ is the Euler–Poincaré characteristic of the sheaf F.

Proof. Since X is normal, we have $R^0\rho_*\mathcal{O}_Z = \mathcal{O}_X$. On the other hand $\chi(R^1\rho_*\mathcal{O}_Z) = \sum p_g(X,x)$ by definition. Thus by the Leray spectral sequence we get the first equality. As for the second one we first note that $h^2(\mathcal{O}_X) = h^0(\omega_X) = h^0(\mathcal{O}_X) = 1$ by the Serre-Grothendieck duality. We have by Lemma 1.1 that $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1-0+1 = 2$. Here we denote $h^i(F) = \dim_{\mathbb{C}} H^i(F)$. Q.E.D.

Lemma 1.3. There exists an effective divisor D on Z with $\omega_Z \cong \mathcal{O}_Z(-D)$. Moreover

$$\operatorname{Supp} D = \bigcup_{x \in X:\text{singular points with } p_g(X,x)>0} \rho^{-1}(x).$$

Proof. Let $x \in X$ be the one of the singular points and $U \subset X$ be its sufficiently small neighbourhood. Set $V = \rho^{-1}(U)$. Let $\rho^{-1}(x) = \bigcup_{i=1}^{n} A_i$ be the decomposition of the exceptional curve into irreducible curves. Let $\phi\in\Gamma(U,\omega_U)$ be a section not vanishing on U. Then $\rho^*\phi$ defines a rational two form on V. Thus there exist intagers $a_i\in\mathbb{Z}$ with $\omega_V \cong \mathcal{O}_V(\Sigma a_i A_i)$. Now recall that the intersection matrix $(A_i\cdot A_j)_{1\leqq i,j\leqq n}$ is negative definite. In particular $-A_i^2>0$. By adjunction formula we have

$$\omega_V\cdot A_i = 2p_a(A_i)-2-A_i^2. \qquad (*)$$

If the arithmetic genus $p_a(A_i)\geqq 1$, then the value of $(*)$ is positive. In case of $p_a(A_i) = 0$, $A_i^2 \leqq -2$ since $\rho^{-1}(x)$ containes no exceptional curve of the first kind by the minimality of ρ. Anyway one sees that $(*)$ is non-negative. It follows easily from this fact that $a_i\leqq 0$ for every i. Since $p_g(X,x) = \dim_{\mathbb{C}}\Gamma(V-\bigcup A_i,\ \omega_V)/\Gamma(V,\omega_V)$, (Cf. Laufer [11]) the condition $a_1 = a_2 = \quad = a_n = 0$ is equivalent to that $p_g(X,x) = 0$. Assume that there exists i with $a_i<0$. We show that $a_j<0$ for every j under this assumption. If for some j, $a_j = 0$, then there exists k with $a_k = 0$ and $\omega_V A_k = -\Sigma(-a_\ell)A_\ell\cdot A_k>0$ since $\bigcup A_i$ is connected, which is a contradiction. Considering all singular points on X we obtain the lemma since $\omega_X \cong \mathcal{O}_X$. Q.E.D.

Proposition 1.4. Let X be a normal quartic surface in $\mathbb{P}^3$. Set $P = \sum_{x\in X:\text{singular points}} p_g(X,x)$.

<1> If $P = 0$, then the minimal resolution Z of X is a K3 surface.

<2> If $P = 1$, then Z is a rational surface with an anti-canonical effective divisor D.

<3> If $P \geqq 2$, then Z is birationally equivalent to a ruled surface over a smooth irreducible curve of genus $P-1$.

Proof. If $P = 0$, $\omega_Z \cong \mathcal{O}_Z$ by Lemma 1.3 and $R^1 \rho_* \mathcal{O}_Z = 0$. By the Leray spectral sequence and Lemma 1.1 we have $H^1(\mathcal{O}_Z) = 0$. Thus Z is a K3 surface.

Assume $P = 1$. By Lemma 1.3 one sees that $\omega^{\otimes m} \cong \mathcal{O}_Z(-mD)$ for an effective divisor $D \neq 0$. In particular the Kodaira dimension $\kappa(Z)$ of Z is $-\infty$. By the theory of classification of surfaces (Cf. Shafarevich [19]) one sees that Z is birationally equivalent to $\mathbb{P}^2$ or a ruled surface over a curve with positive genus. On the other hand we have $\chi(\mathcal{O}_Z) = 2-P$ by Lemma 1.2. Since the Euler-Poincaré characteristic of the structure sheaf is a birational invariant, one sees that Z is rational.

In the case where $P \geqq 2$, we have <3> by the same reason.

Q.E.D.

Remark. In Umezu [21] Y. Umezu showed that if $P \geqq 2$, then $P = 2$ or 4 and she gave the classification of quartic surfaces with $P \geqq 2$.

Next we consider sextic curves. Let B be a reduced sextic curve (i.e., a curve of degree 6 with no multiple components) in the 2 dimensional projective space $\mathbb{P}^2$. We introduce the branched double covering X of $\mathbb{P}^2$ branching along B. Let $F(z_0, z_1, z_2)$ be the homogeneous defining polynomial of B. We give weight 1, 1, and 1 to z_0, z_1 and z_2 respectively. Let z_3 be another variable with weight 3. Then $z_3{}^2-F(z_0,z_1,z_2)=0$ defines a surface X in the weighted projective space $\mathbb{P}(1,1,1,3)$ not passing through the point $(0,0,0,1)$. (The quotient of $\mathbb{C}^4-\{(0,0,0,0)\}$ by the following action of $\mathbb{C}^*=\mathbb{C}-\{0\}$ is $\mathbb{P}(1,1,1,3)$. Action: $t(z_0,z_1,z_2,z_3)=(tz_1,tz_2,tz_3,t^3z_3)$ where $t\in\mathbb{C}^*$ and $(z_0,z_1,z_2,z_3)\in\mathbb{C}^4-\{(0,0,0,0)\}$. $\mathbb{P}(1,1,1,3)$ has a unique singular point at $(0,0,0,1)$.) The restriction to X of the projection $\pi\colon \mathbb{P}(1,1,1,3)-\{(0,0,0,1)\}\longrightarrow \mathbb{P}^2$, $(z_0,z_1,z_2,z_3)\longmapsto(z_0,z_1,z_2)$ defines a finite morphism of degree 2. We denote it by the same letter $\pi\colon X\longrightarrow \mathbb{P}^2$. The following lemma is easily checked. (Cf. Arnold [1])

Lemma 1.5. A point $x\in X$ is singular if and only if $\pi(x)$ is a singular point of B. Moreover the isomorphism class of a surface singularity (X, x) and that of a curve singularity $(B, \pi(x))$ determine each other uniquely. Thus singular points on X and those on B has one-to-one correspondence.

Lemma 1.6. For a branched double covering X branching along a sextic curve B, we have:

(1) The dualizing sheaf ω_X is trivial, i.e., $\omega_X\cong\mathcal{O}_X$.

(2) $H^1(\mathcal{O}_X)=0$.

Proof. (1) Let L be a general line in $\mathbb{P}^2$. We have

$$\omega_X\cong\pi^*\omega_{\mathbb{P}^2}(\tfrac{1}{2}\pi^*B)\cong\pi^*\mathcal{O}_{\mathbb{P}^2}(-3L)\otimes\mathcal{O}_{\mathbb{P}^2}(3L)\cong\mathcal{O}_X.$$

(2) For every point $p \in \mathbb{P}^2$, we have $f \in \mathcal{O}_{\mathbb{P}^2,p}$ such that

$$(\pi_*\mathcal{O}_X)_p \cong \mathcal{O}_{\mathbb{P}^2,p}[z]/(z^2-f)$$

where z is an indeterminate. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \pi_*\mathcal{O}_X \longrightarrow M \longrightarrow 0$$

where M is a line bundle on $\mathbb{P}^2$. Since $H^1(\mathcal{O}_{\mathbb{P}^2}) \cong H^1(M) = 0$, one sees that $H^1(\pi_*\mathcal{O}_X) = 0$. By the Leray spectral sequence we have $H^1(\mathcal{O}_X) = 0$ since $R^q\pi_*\mathcal{O}_X = 0$ for $q>0$. Q.E.D.

Once we establish Lemma 1.6, by the very same reason as quartic surfaces, we can show the following proposition.

Proposition 1.7. Let X be a branched double covering of $\mathbb{P}^2$ branching along a reduced sextic curve B. Let $\rho: Z \longrightarrow X$ be the minimal resolution of singularities. Set

$$P = \sum_{x \in X:\text{singular points}} p_g(X,x).$$

<1> If $P = 0$, then Z is a K3 surface.

<2> If $P = 1$, then Z is a rational surface with an anti-canonical effective divisor D.

<3> If $P \geq 2$, then Z is birationally equivalent to a ruled surface over a smooth irreducible curve with genus $P-1$.

Remark. In section 6 we show that if $P \geq 2$, then $P = 2$ or 3.

According to Lemma 1.5 we can study X instead of B. We take up mainly in this article case <2> in Proposition 1.4 and case <2> in Proposition 1.7.

Let X be a normal quartic surface or a branched double covering branching along a reduced sextic curve. Assume that X has unique $\tilde{E}_8$ singularity plus several rational double points and no other singularities. The minimal resolution Z of X is rational with a non-zero effective anti-canonical divisor D. Moreover in this case D is an irreducible smooth elliptic curve with self-intersection number $D^2 = -1$. If X has $T_{2,3,7}$ instead of $\tilde{E}_8$, then D is an irreducible rational curve with one ordinary double point with self-intersection number $D^2 = -1$. If X has E_{12} instead of $\tilde{E}_8$, then D is an irreducible rational curve with one ordinary cusp with $D^2 = -1$.

Proposition 1.8. Assume that Z is a smooth rational surface with an effective irreducible anti-canonical divisor D. If Z is not a relatively minimal model, then Z can be blown-down to $\mathbb{P}^2$.

Proof. Since any relatively minimal rational surface is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or a Hirzebruch surface Σ_k with $k \geqq 2$, Z can be blown-down to one of them.

<u>Case 1</u>. Assume that there exists a birational morphism $\sigma : Z \longrightarrow \Sigma_k$. Since $\Sigma_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$, there exist smooth rational curves Δ, F on Σ_k with $\Delta^2 = -k$, $F^2 = 0$ and $F \cdot \Delta = 1$. First we note that $\sigma(D)$ is a member of the anti-canonical linear system $|-\omega_{\Sigma_k}|$

of Σ_k since $\sigma_* \omega_Z = \omega_{\Sigma_k}$. By the adjunction formula we have

$$0 = p_a(\Delta) = (\Delta^2 - \sigma(D)\ \Delta)/2+1 = -(k+\sigma(D)\cdot\Delta)/2+1.$$

It implies $\sigma(D) \neq \Delta$ and thus $k = 2$, $\sigma(D)\cdot\Delta = 0$. Now since Z is not a relatively minimal model, σ is decomposed into two morphisms $\sigma = \sigma' \circ \sigma''$, where $\sigma'\colon \Sigma' \longrightarrow \Sigma_2$ is a blowing-up of a poit $p \in \Sigma_2$ and $\sigma''\colon Z \longrightarrow \Sigma'$ is a birational morphism. If $p \notin \sigma(D)$, then $D \notin |-\omega_Z|$. Thus $p \in \sigma(D)$ and $p \notin \Delta$ since $\sigma(D) \cap \Delta = \phi$. Let F_p be a smooth rational curve on Σ_2 passing through p with ${F_p}^2 = 0$. Let F' and Δ' be the strict inverse image by σ' of F_p and Δ respectively. F' is an exceptional curve of the first kind on Σ'. Let $\sigma_1\colon \Sigma' \longrightarrow \Sigma^{(2)}$ be the contraction of F'. Then $\sigma_1(\Delta')$ is an exceptional curve of the first kind on $\Sigma^{(2)}$. Let $\sigma_2\colon \Sigma^{(2)} \longrightarrow \Sigma^{(3)}$ be its contraction. Set $\tilde{\sigma} = \sigma_2 \sigma_1 \sigma''\colon Z \longrightarrow \Sigma^{(3)}$. Since ${\omega_{\Sigma_2}}^2 = 8$, we have ${\omega_{\Sigma^{(3)}}}^2 = 9$, which implies $\Sigma^{(3)} \cong \mathbb{P}^2$. Thus $\tilde{\sigma}$ defines a blowing-down to $\mathbb{P}^2$.

Figure 1.1

<u>Case 2</u>. Assume that there exists a birational morphism $\sigma\colon Z \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Now since Z is not a relatively minimal one, σ is decomposed into two morphisms $\sigma = \sigma' \circ \sigma''$, where $\sigma'\colon \Sigma' \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a blowing-up of a point $p \in \mathbb{P}^1 \times \mathbb{P}^1$ and $\sigma''\colon Z \longrightarrow \Sigma'$ is a birational morphism. We have a smooth rational curve F and G on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through p with $F^2 = G^2 = 0$ and $F \cdot G = 1$. Let F' and G' be the strict inverse image of F and G by σ' respectively. F' and G' are the exceptional curves of the first kind and

they are mutually disjoint. Let $\sigma_1: \Sigma' \longrightarrow \Sigma$ be the contraction of F' and G'. Set $\tilde{\sigma} = \sigma_1 \circ \sigma': Z \longrightarrow \Sigma$. Since $\omega_{\mathbb{P}^1 \times \mathbb{P}^1}{}^2 = 8$, we have $\omega_\Sigma{}^2 = 9$, which implies that $\Sigma \cong \mathbb{P}^2$.

Consequently in any case there exists a birational morphism $\sigma: Z \longrightarrow \mathbb{P}^2$. Q.E.D.

Corollary 1.9. A non-zero irreducible anti-canonical effective divisor on a smooth rational surface Z is either;

(a) an irreducible smooth elliptic curve

(b) an irreducible rational curve with one ordinary double point.

or (c) an irreducible rational curve with one ordinary cusp.

In particular examples taken up just before Proposition 1.8 exhaust all the possibilities.

Proof. First assume that Z is not a minimal model. By Proposition 1.8, there exists a birational morphism $\sigma: Z \longrightarrow \mathbb{P}^2$. Since every birational morphism between surfaces is a composition of blowing-ups, we can write $\sigma = \sigma' \circ \sigma_1$ where $\sigma_1: Z \longrightarrow X'$ is a blowing-up of a point $x' \in X'$ on a smooth surface X' and $\sigma': X' \longrightarrow \mathbb{P}^2$ is a birational morphism. By induction on the number of blowing-ups, we can assume that $D' = \sigma_1(D)$ is one of above (a), (b), (c) since $D' \in |-\omega_{X'}|$. If $x' \notin D'$, then we have $D \notin |-\omega_Z|$, a contradiction. Thus $x' \in D'$. Let m be the multiplicity of x' as a point of D'. Since $D+(m-1)\sigma_1^{-1}(x')\in|-\omega_Z|$, one knows that $m = 1$, i.e., x' is a simple point of D'. Thus σ_1 induces an isomorphism $\sigma_1\colon D \longrightarrow D'$ and D is one of (a),(b),(c).

If Z is a minimal model, then by the proof of Proposition 1.8, Z is isomorphic to either $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ or Σ_2. Moreover according to the proof of Proposition 1.8, there exists a birational map $\sigma'\colon Z \dashrightarrow \mathbb{P}^2$ such that its restriction $\sigma'|D$ to D is an isomorphism. Thus we complete the proof. Q.E.D.

§ 2. A theorem of Torelli type.

In this section, we would like to explain a theorem of Torelli type for rational surfaces with an effective anti-canonical divisor. Most of the essential ideas of this theorem are due to Looijenga. However the situation we treat here is a bit different from Looijenga's original one. (Looijenga [10])

Because the proof of the theorem is the same as the one we gave in [15], we omit it.

Though in [15] we used a lemma due to Demazure which treats the case where the self-intersection number $\omega_Z{}^2$ of the dualizing sheaf is positive, our proof in [15] is valid without any change because Looijenga verified in his recent work [10] the same lemma for the case $\omega_Z{}^2 \leq 0$. (To be precise the situation Looijenga treated is a bit different from ours in this article. However his proof is valid without any change.)

Anyway we would like to begin this section by explaining several notions. —— Dynkin graphs, Weyl groups, roots, etc.

Let Z be a smooth rational surface with irreducible effective anti-canonical divisor D. Moreover we assume in this section that the self-intersection number of the dualizing sheaf $\omega_Z{}^2$ is less than or equal to 6. Set $t = 9-\omega_Z{}^2$. We have $t \geq 3$. Under this assumption, Z is not a minimal model. Thus by Proposition 1.8, we have a sequence

(2.1) $\quad Z = Z_t \xrightarrow{\sigma_t} Z_{t-1} \xrightarrow{\sigma_{t-1}} \quad \longrightarrow Z_2 \xrightarrow{\sigma_2} Z_1 \xrightarrow{\sigma_1} Z_0 = \mathbb{P}^2$

where each σ_i is a blowing-up of a point $z_i \in Z_{i-1}$. We denote

$D_t = D$, $D_{i-1} = \sigma_i(D_i)$ $(1 \leq i \leq t)$. We have $z_i \in D_{i-1} \subset Z_{i-1}$. We consider the Picard group $\mathrm{Pic}(Z)$. Let e_0 be the class of the total inverse image on Z of a line in $Z_0 = \mathbb{P}^2$. Let e_i $(i \geq 1)$ be the class of the total inverse image on Z of the exceptional curve $\sigma_i^{-1}(z_i)$. Elements $e_0, e_1, \cdot\ e_t \in \mathrm{Pic}(Z)$ defines a free $\mathbb{Z}$-basis with the following mutual intersection numbers;

$$e_0^2 = +1,\ e_i^2 = -1\ (1 \leq i \leq t),\ e_i\ e_j = 0\ (i \neq j).$$

We say that (2.1) is the <u>blowing-down</u> <u>sequence</u> along $e_0, e_1, \cdot e_t$, when each e_i is the above-mentioned class of effective divisors. Here we note that

$$\omega_Z = \mathcal{O}_Z(-D) = -3e_0 + e_1 + \cdot\ \cdot + e_t.$$

Let $P = \mathbb{Z}\varepsilon_0 + \mathbb{Z}\varepsilon_1 + \quad + \mathbb{Z}\varepsilon_t$ be a $\mathbb{Z}$-module with a bilinear form which is isomorphic to $\mathrm{Pic}(Z)$ with the intersection form, where $\varepsilon_0, \quad , \varepsilon_t \in P$ is a basis with

$$\varepsilon_0^2 = +1,\ \varepsilon_i^2 = -1\ (1 \leq i \leq t),\ \varepsilon_i\ \varepsilon_j = 0\ (i \neq j).$$

We set $\kappa = -3\varepsilon_0 + \varepsilon_1 + \quad + \varepsilon_t$. Let Γ be the orthogonal complement of $\mathbb{Z}\kappa$ in P. $\Gamma = \{\ x \in P \mid x \cdot \kappa = 0\ \}$. The restriction of the bilinear form of P to Γ is described by the following graph.

Here we denote $\tau_1 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, $\tau_j = \varepsilon_{j-1} - \varepsilon_j$ $(2 \leq j \leq t)$ for simplicity. Vertices o corresponding to τ_i indicates a member of a basis of Γ with the self-intersection -2. (It is easily checked that the above $\tau_1, \tau_2, \quad , \tau_t$ defines a basis of Γ and that $\tau_i^2 = -2$ if $t \geq 3$.) Two vertices $\overset{\tau_i}{o}$ $\overset{\tau_j}{o}$ are connected with an edge —— if $\gamma_i \gamma_j = 1$ and they are not connected if $\gamma_i \cdot \gamma_j = 0$. In particular Γ is isomorphic to the root lattice (Cf. Bourbaki [3]) of type A_2+A_1, A_4, D_5, E_6, E_7 or E_8 according as $t = 3, 4, 5, 6, 7, 8$. If $t \geq 9$, then Γ is not negative-definite.

Let $\gamma \in P$ be an element with $\gamma^2 = -2$. Let $s_\gamma : P \longrightarrow P$ be a linear map defined by $s_\gamma(x) = x+(x\cdot\gamma)\gamma$ for $x \in P$. It is easily checked that s_γ is an isomorphism of order 2 preserving the bilinear form. In addition if $\gamma \kappa = 0$, then $s_\gamma(\kappa) = \kappa$. s_γ is called the <u>reflection</u> associated to γ. The group generated by $s_{\gamma_1}, \cdots, s_{\gamma_t}$ is called the <u>Weyl group</u> of P and it is denoted by W or W_P. (For $w \in W$ $w(\kappa) = \kappa$) We call an element in $\bigcup_{i=1}^{t} W\gamma_i$ ($\subset \Gamma$) a <u>root</u>.

Indeed s_γ defines the reflection with respect to the hyperplane orthogonal to γ i.e., $\{ x \in P\otimes\mathbb{R} \mid x\cdot\gamma = 0 \}$ in $P\otimes\mathbb{R}$. $(W, \{s_{\gamma_1}, s_{\gamma_2}, \cdots, s_{\gamma_t}\})$ defines a Coxeter system. (Cf. Looijenga [10], Bourbaki [3]) Now let $\gamma \in \Gamma$ be a root. Writing $\gamma = \sum_{i=1}^{t} n_i \gamma_i$ $(n_i \in \mathbb{Z})$, then we have either $n_i \geq 0$ for any i or $n_i \leq 0$ for any i. If $n_i \geq 0$ for any i, we say that γ is a <u>positive root</u>. Otherwise it is called a <u>negative root</u>. Note that this notion depends on the choice of the basis. Let $R_+(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_t)$ denote the set of positive roots.

For roots in Pic(Z) we can distinguish the following property. A root $r \in \mathrm{Pic}(Z)$ is called a <u>nodal root</u> if the restriction of r to D is a trivial line bundle. (This terminology is due to Looijenga.)

Lemma 2.1. Let $r \in \mathrm{Pic}(Z)$ be a nodal root. Then either r or $-r$ is effective.

Proof. Assume that $r^2 = -2$, $r|_D \cong \mathcal{O}_D$ and $H^0(-r) = 0$. By Serre duality we have $H^2(r(-D)) = 0$. Consider the exact sequence

$$0 \longrightarrow r(-D) \longrightarrow r \longrightarrow r|_D \longrightarrow 0.$$

One sees that $h^2(r) = 0$ and $H^1(r) \longrightarrow H^1(r|_D) \cong \mathbb{C}$ is surjective. Thus $h^1(r)>0$. By Riemann-Roch formula

$$h^0(r) = (r^2+D\cdot r)/2+1+h^1(r)>0,$$

i.e., r is effective. Q.E.D.

Let S_+ denote the set of effective nodal roots. $S = S_+ \cup (-S_+)$ is the set of nodal roots. Let W_S be the group generated by $\{ s_r \mid r \in S \}$. W_S is a subgroup of $W_{\mathrm{Pic}(Z)}$. We call W_S the Weyl group of Z associated to nodal roots.

The following theorem is due to Demazure when $3 \leqq t \leqq 9$ and it is due to Looijenga when $t \geqq 10$. (Though the situation they treated is a bit different from ours, their proof is valid without any change.) (Demazure [5], Looijenga [10])

Theorem 2.2. Let Z be a rational surface with an effencive irreducible anti-canonical divisor D such that $t = 9-\omega_Z{}^2 \geqq 3$. Let e_0, e_1, , $e_t \in \mathrm{Pic}(Z)$ be a basis such that there exists a blowing-down sequence along $e_0, e_1, \quad , e_t$. Let W be the Weyl group of $Pic(Z)$ defined depending on $e_0, e_1, \quad , e_t$ and let $w \in W$. Then there exists a blowing-down sequence along $w(e_0), w(e_1), \quad , w(e_t)$ if and only if every effective nodal root is a positive root, i.e., $S_+ \subset R_+(w(e_0), w(e_1), \quad , w(e_t))$. Moreover for any two basis e_0, $e_1, \quad , e_t \in Pic(Z)$ and $e_0', e_1', \cdot \quad , e_t' \in Pic(Z)$ such that there exist blowing-down sequences along both of them, there exists an element $w \in W$ with $e_i' = w(e_i)$ for $0 \leq i \leq t$.

Corollary 2.3. The set of roots R in $Pic(Z)$ and the Weyl group W of $Pic(Z)$ do not depend on the choice of the blowing-down sequence (2.1).

Note that the positive cone $\{x \in P \otimes \mathbb{R} \mid x \cdot x > 0\}$ in $P \otimes \mathbb{R}$ has two connected component since the signature of the bilinear form of P is $(1, t)$.

Definition 2.4. Let t be an integer with $t \geq 3$. Let E be a one-dimensional algebraic group isomorphic to a smooth elliptic curve, $\mathbb{C}^* = \mathbb{C} - \{0\}$, or $\mathbb{C}$. We call the following object $\underline{Z} = (Z, D, \alpha, \iota)$ a marked rational surface over E of degree $9-t$.

(1) The first item Z is a smooth rational surface with $\omega_Z^2 = 9-t$.

(2) The second item D is an effective irreducible anti-canonical divisor on Z which has the following isomorphism ι.

(3) The third one $\alpha: P \longrightarrow \mathrm{Pic}(Z)$ is a linear isomorphism satisfying the following conditions (i), (ii), (iii) and (iv), where $P = \mathbb{Z}\varepsilon_0 + \mathbb{Z}\varepsilon_1 + \quad + \mathbb{Z}\varepsilon_t$ is an abstract free $\mathbb{Z}$-module with a bilinear form defined by $\varepsilon_0^2 = +1$, $\varepsilon_i^2 = -1$ $(1 \leqq i \leqq t)$, $\varepsilon_i\ \varepsilon_j = 0$ $(i \neq j)$.

(i) α preserves the bilinear form, i.e., $x \cdot y = \alpha(x) \cdot \alpha(y)$ for any $x, y \in P$.

(ii) $\alpha(\kappa) = \omega_Z$ where $\kappa = -3\varepsilon_0 + \varepsilon_1 + \cdot \quad + \varepsilon_t$.

(iii) $\alpha(\Pi) = R$ where Π and R are the sets of roots in P and $\mathrm{Pic}(Z)$ respectively.

(iv) $\alpha(\Lambda_+) = C_+$ where Λ_+ (resp. C_+) is a connected component of the positive cone in $P \otimes \mathbb{R}$ (resp. $\mathrm{Pic}(Z) \otimes \mathbb{R}$) containing ε_0. (resp. e_0)

(4) The fourth one $\iota: \mathrm{Pic}^0(D) \longrightarrow E$ is an isomorphism as algebraic groups, where $\mathrm{Pic}^0(D)$ is the connected component of $\mathrm{Pic}(D)$ containing the zero element.

Definition 2.5. Two marked rational surface over E (Z, D, α, ι) and $(Z', D', \alpha', \iota')$ are <u>isomorphic</u> if there exists an isomorphism of varieties $f: Z \longrightarrow Z'$ satisfying the following conditions (A), (B), and (C).

(A) $f(D) = D'$.

(B) The composition

$$\mathrm{Pic}(Z) \xleftarrow[\sim]{\alpha} P \xrightarrow{\alpha'} \mathrm{Pic}(Z') \xrightarrow{f^*} \mathrm{Pic}(Z)$$

can be written as a composition of finite reflections corresponding to nodal roots on Z.

(C) The diagram
$$\begin{tikzcd} \mathrm{Pic}^0(D') \arrow[rr, "f^*"] \arrow[dr, "\iota'"'] & & \mathrm{Pic}^0(D) \arrow[dl, "\iota"] \\ & E & \end{tikzcd}$$
is commutative.

Definition 2.6. Let $Q \subset \mathrm{Pic}(Z)$ be the orthogonal complement of $\mathbb{Z}\omega_Z$, i.e., $Q = \{\, x \in \mathrm{Pic}(Z) \mid x \cdot \omega_Z = 0 \,\}$. Note that the image of Q by the restriction map $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(D)$ is contained in $\mathrm{Pic}^0(D)$. The following composition of homomorphisms is called the <u>characteristic homomorphism</u> $\phi_{\underline{Z}}$ of $\underline{Z} = (Z, D, \alpha, \iota)$.

$$\Gamma \xrightarrow{\alpha} Q \xrightarrow{\text{restriction}} \mathrm{Pic}^0(D) \xrightarrow{\iota} E$$

Here Γ is the orthogonal complement of $\mathbb{Z}\kappa$ in P.

It is easy to check the next lemma.

Lemma 2.7. The characteristic homomorphism $\phi_{\underline{Z}}$ depends only on the isomorphism class of (Z, D, α, ι).

Now we can state the main theorem in this section. It gives a powerful tool to study rational surfaces. Even though the situation treated by Looijenga is a bit different from ours, this theorem is due to Looijenga, we think.

Theorem 2.8. (A theorem of Torelli type.) The map induced by associating a marked rational surface (Z, D, α, ι) to its characteristic homomorphism

$$\underline{\{(Z,\ D,\ \alpha,\ \iota):\ \text{a marked rational surface over } E \text{ of degree } 9-t\}}_{\text{isomorphisms}} \longrightarrow \mathrm{Hom}(\Gamma,\ E)$$

is bijective.

Next we would like to explain why this theorem is called one of Torelli type. It is explained by the Deligne's mixed Hodge theory. For simplicity we assume that D is an irreducible smooth elliptic curve with $D^2 = -1$. Consider an exact sequence of mixed Hodge structures (Cf. Deligne [4])

$$H^0(D)(-1) \longrightarrow H^2(Z) \longrightarrow H^2(Z-D) \longrightarrow H^1(D)(-1).$$

Note that $W_2H^2(Z) = H^2(Z)$, $W_1H^2(Z) = 0$, $F^1H^2(Z) = H^2(Z)$, $F^2H^2(Z) = 0$, $W_3(H^1(D)(-1)) = H^1(D)$, $W_2(H^1(D)(-1)) = 0$, $F^1(H^1(D)(-1)) = H^1(D)$, $F^2(H^1(D)(-1)) = H^0(\omega_D)$, and $F^3(H^1(D)(-1)) = 0$. Thus we know that $\dim_{\mathbb{C}} F^2H^2(Z-D) = 1$. Now by definition $F^2H^2(Z-D)$ is represented by a logarithmic 2-form ϕ on Z with the pole along D, which is unique up to constant multiple. Since this situation is very similar to that of the second cohomology group of K3 surfaces, we can consider the periods of ϕ. Here the periods are nothing but the linear mapping

$$H_2(Z-D) \longrightarrow \mathbb{C};\quad \Delta \longrightarrow \int_\Delta \phi\ .$$

Note that there is a submodule $\mathrm{Im}\ (H_1(D) \xrightarrow{\tau} H_2(Z-D))$. Since

$\int_{\tau(\gamma)}\phi = 2\pi\sqrt{-1}\int_\gamma \mathrm{Res}(\phi)$, we have that

$\mathbb{C}/\mathrm{Im}(H_1(D) \longrightarrow H_2(Z-D) \longrightarrow \mathbb{C}) \cong D$. Let Q be a orthogonal complement of $\mathbb{Z}\omega_Z$ in $\mathrm{Pic}(Z)$. One sees easily that there exists an exact sequence

$$0 \longrightarrow H_1(D) \longrightarrow H_2(Z-D) \longrightarrow Q \longrightarrow 0.$$

Thus we have an induced group homomorphism $Q \longrightarrow D$. We can check that this homomorphism is identified with the restriction of the mapping $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(D)$. Therefore the characteristic homomorphism $\phi_{\underline{Z}}$ can be regarded as the periods of $Z-C$. This is the reason why the above theorem is called one of Torelli type.

§ 3. Properties of line bundles

This section is devoted to study properties of line bundles on a smooth rational surface Z with an effective irreducible anti-canonical divisor D. We owe ideas in this section greatly to Saint-Donat [17].

Recall that a line bundle L (resp. a divisor C) on Z is <u>numerically effective</u> if for any curve A on Z, the intersection $L\cdot A$ (resp. $C\cdot A$) is non-negative.

Definition 3.1. A line bundle L on Z with the following properties are called a <u>polarization</u> of Z.

(1) The self-intersection number L^2 is positive.

(2) L is numerically effective.

(3) The restriction of L to D is a trivial line bundle, i.e., $L|_D \cong \mathcal{O}_D$.

(4) For every exceptional curve of the first kind A, the intersection $L\ A$ is strictly positive. $(L\cdot A>0)$

The number L^2 is called the <u>degree</u> of L.

Lemma 3.2. (1) If Z has a polarization, then $t = 9-\omega_Z{}^2\geqq 10$.

(2) For any polarization L, $h^1(L) = 1$ and $h^0(L) = (L^2/2)+2$. Moreover the linear system $|L|$ has no fixed points on D.

Proof. (1) If $t\leqq 9$, for every element $M\in \mathrm{Pic}(Z)$ with $M\cdot\omega_Z = 0$, $M^2\leqq 0$ holds. However $L^2>0$ and $L\ \omega_Z = 0$ for any polarization.

(2) By the Kawamata-Ramanujam vanishing theorem (Kawamata [9]), we have $H^1(L(-D)) = H^2(L(-D)) = 0$. Thus the mapping $H^0(L) \longrightarrow H^0(L|_D) \cong H^0(\mathcal{O}_Z) \cong \mathbb{C}$ is surjective, and $h^1(L) = h^1(\mathcal{O}_D) = 1$, $h^2(L) = 0$. Surjectivity implies that $|L|$ has no fixed points on D. On the other hand by the Riemann-Roch formula we have

$$h^0(L) = (L^2 - L\,\omega_Z)/2 + \chi(\mathcal{O}_Z) + h^1(L) - h^2(L) = (L^2/2) + 2. \qquad \text{Q.E.D.}$$

If X is a normal quartic surface in $\mathbb{P}^3$ and $\rho: Z \longrightarrow X \subset \mathbb{P}^3$ is its minimal resolution of singularities, then $L = \rho^*\mathcal{O}_{\mathbb{P}^3}(1)$ is a polarization of degree 4. Similarly for a branched double covering branching along a sextic curve we can define a polarization of degree 2. However note that conversely the polarization L does not necessarily defines a generically one-to-one morphism $\Phi_L: Z \longrightarrow \mathbb{P}^N$. The linear system $|L|$ may have fixed components. Even if it has no fixed components, it may have isolated fixed points. Even if it has no fixed points, it may define a morphism whose degree is greater than 1.

In this section we give a necessary and sufficient condition in order that L does not define a generically one-to-one morphism in the case $L^2 = 2$ or 4.

Proposition 3.3. Let M be a line bundle on Z satisfying

(a) $H^0(M) \neq 0$

(b) The linear system $|M|$ has no fixed components. And

(c) the intersection $M \cdot D$ is zero.

(1) If the image of the rational map ϕ_M associated to M is a curve, then $M^2 = 0$.

(2) One of the following (i), (ii) holds.

(i) $M^2>0$, any generic member of $|M|$ is an irreducible curve with arithmetic genus $(M^2/2)+1$ and $h^1(M) = 1$.

(ii) $M^2 = 0$ and there exists a smooth irreducible elliptic curve F and a positive integer k with $M \cong \mathcal{O}_Z(kF)$. Moreover $h^1(M) = k$. Every member of $|M|$ can be written as $F_1+F_2+ \quad +F_k$, where $F_i \in |F|$.

Proof. Firstly assume that the image Γ' of the rational map ϕ_M: $Z \dashrightarrow \mathbb{P}^N$ associated to M is a curve. Let ν: $\Gamma \longrightarrow \Gamma'$ be the normalization of Γ'. For a suitable choice of a birational morphism τ: $\hat{Z} \longrightarrow Z$, there exists a morphism $\hat{\Phi}$: $\hat{Z} \longrightarrow \Gamma$ with $\phi_M \tau = \nu\hat{\Phi}$.

$$\begin{tikzcd} & \hat{Z} \arrow[dl, "\tau"'] \arrow[dr, "\hat{\Phi}"] & \\ Z \arrow[dr, dashed, "\phi_M"'] & & \Gamma \arrow[dl, "\nu"] \\ & \Gamma' & \end{tikzcd}$$

If the genus of Γ is positive, we have a non-zero global regular 1-form α on Γ. Since $\hat{\Phi}^*\alpha$ defines a non-zero global regular 1-form on $\hat{Z}$, we have $H^0(\Omega^1_{\hat{Z}}) \neq 0$, which contradicts to that $\hat{Z}$ is rational. Thus Γ is a smooth rational curve. It implies that for every point p, $p' \in \Gamma$, divisors $\tau(\hat{\Phi}^{-1}(p))$ and $\tau(\hat{\Phi}^{-1}(p'))$ are linearly equivalent. Choose a general point $q \in \Gamma$ and set $F = \tau(\hat{\phi}^{-1}(q))$. One sees that $M \cong \mathcal{O}_Z(kF)$ for some integer k. If $\dim|F| \geq 2$, then we have a member $F_1 \in |F|$ such that for any point $p \in \Gamma$, $F_1 \neq \tau(\hat{\phi}^{-1}(p))$. Choose points $q = q_1, q_2, \quad , q_k \in \Gamma$ such that $\tau(\hat{\phi}^{-1}(q_1)) + \tau(\hat{\phi}^{-1}(q_2)) + \quad + \tau(\hat{\phi}^{-1}(q_k)) \in |M|$. Since $\tau(\hat{\phi}^{-1}(q_1)) = F \sim F_1$, we have $G = F_1 + \tau(\hat{\phi}^{-1}(q_2)) + \quad + \tau(\hat{\phi}^{-1}(q_k)) \in |M|$ since $|M|$ is a complete linear system. However, by the choice of F_1 and the definition of $\hat{\phi}$, we have $G \notin |M|$, a contradiction. Therefore we have $\dim |F| = 1$.

We have $kD \cdot F = D \cdot M = 0$ and thus $D \cdot F = 0$. We can conclude that $D \cap F = \phi$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-F-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_F \oplus \mathcal{O}_D \longrightarrow 0.$$

It implies $h^1(\mathcal{O}_Z(-F-D)) = 1$. By the Serre duality, we have $h^1(\mathcal{O}_Z(F)) = 1$. Moreover $h^2(\mathcal{O}_Z(F)) = h^0(\mathcal{O}_Z(-F-D)) = 0$. It follows from the Riemann-Roch formula

$$\begin{aligned} 2 = 1+\dim|F| &= h^0(\mathcal{O}_Z(F)) \\ &= \chi(\mathcal{O}_Z)+(F^2+F\cdot D)/2+h^1(\mathcal{O}_Z(F)) = F^2/2+2 \end{aligned}$$

that $F^2 = 0$ and $M^2 = k^2F^2 = 0$. In particular the linear system $|F|$ has no fixed points and F is smooth by the Bertini theorem. By adjunction formula F is an elliptic curve.

Next we would like to compute $h^1(M)$. Let $F_1, F_2, \quad , F_k \in |F|$ be general members. We can assume that $F_1, \quad , F_k$ and D are mutually disjoint since $D \cdot F = 0$ and $F^2 = 0$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-F_1- \quad -F_k-D) \longrightarrow \mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{F_i} \oplus \mathcal{O}_D \longrightarrow 0$$

and the Serre duality, one sees that $h^1(\mathcal{O}_Z(F_1+ \quad +F_k)) = h^1(M) = k$.

Secondly assume that the image of ϕ_M is not a curve. We have $A^2 \geq 0$ since $|M|$ has no fixed components. If $A^2 = M^2 = 0$, then $|M|$ has no fixed points and the image of the morphism ϕ_M is a curve. Thus $A^2 = M^2 > 0$. By the Bertini theorem A is irreucible. We have $p_a(A) = A^2/2+1$ by the adjunction formula. It follows that $A \cap D = \phi$ from $M \cdot D = A \cdot D = 0$. Thus

$$0 \longrightarrow \mathcal{O}_Z(-A-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_A \oplus \mathcal{O}_D \longrightarrow 0$$

is exact and one sees that $h^1(\mathcal{O}_Z(A)) = 1$. Q.E.D.

Lemma 3.4. Let C be an effective divisor on Z with $\mathrm{Supp} C \cap D = \phi$ and $h^0(\mathcal{O}_C) = 1$. Then we have $h^1(\mathcal{O}_Z(C)) = 1$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-C-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_D \longrightarrow 0.$$

We have $h^1(\mathcal{O}_Z(-C-D)) = 1$. By the Serre duality we have the conclusion. Q.E.D.

Lemma 3.5. Let Δ be a non-zero effective divisor on Z with $h^0(\mathcal{O}_Z(\Delta)) = 1$ and $\mathrm{Supp}\,\Delta \cap D = \phi$. We have $h^1(\mathcal{O}_Z(\Delta)) \geq 1$ and $\Delta^2 = -2h^1(\mathcal{O}_Z(\Delta)) \leq -2$.

Proof. Consider the sequence

$$0 \longrightarrow \mathcal{O}_Z(-\Delta-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_\Delta \oplus \mathcal{O}_D \longrightarrow 0.$$

By assumption $\mathrm{Supp}\,\Delta \cap D = \phi$, it is exact. We have $h^1(\mathcal{O}_Z(\Delta)) = h^1(\mathcal{O}_Z(-\Delta-D)) = h^0(\mathcal{O}_\Delta) \geq 1$ since $h^0(\mathcal{O}_Z) = h^0(\mathcal{O}_D) = 1$, $h^1(\mathcal{O}_Z) = 0$. Note that $h^2(\mathcal{O}_Z(\Delta)) = h^0(\mathcal{O}_Z(-\Delta-D)) = 0$. By the Riemann-Roch theorem, we have

$$
\begin{aligned}
1 = h^0(\mathcal{O}_Z(\Delta)) &= \chi(\mathcal{O}_Z)+(\Delta^2+D\cdot\Delta)/2+h^1(\mathcal{O}_Z(\Delta))-h^2(\mathcal{O}_Z(\Delta)) \\
&= 1+(\Delta^2/2)+h^1(\mathcal{O}_Z(\Delta)). \qquad \text{Q.E.D.}
\end{aligned}
$$

Corollary 3.6. Let Θ be an irreducible curve on Z with $h^0(\mathcal{O}_Z(\Theta)) = 1$ and $\Theta\cdot D = 0$. We have $\Theta^2 = -2$ and Θ is a smooth ratioal curve.

Proof. Since Θ and D are irreducible, $\Theta\cdot D = 0$ implies $\Theta \cap D = \phi$. Obviously $h^0(\mathcal{O}_\Theta) = 1$. Thus by Lemma 3.4 and Lemma 3.5, we obtain $\Theta^2 = -2$. Moreover by the adjunction formula, Θ is smooth and rational. Q.E.D.

Proposition 3.7. Let L be a polarization on Z. If $|L|$ has a fixed component, then $|L|$ contains a divisor with the following form; $kF+\Gamma$ where F is an irreducible smooth elliptic curve on Z with $F^2 = 0$ and $D\cdot F = 0$, Γ is an irreducible smooth rational curve with $\Gamma^2 = -2$, $\Gamma\cdot D = 0$ and $\Gamma\cdot F = 1$ and k is an integer with $k \geq 2$. The divisor Γ is the fixed part of $|C|$.

Proof. The proof is a bit complicated. By Lemma 3.2 the linear system $|L|$ is non-empty. Let $C \in |L|$ be a general member. Let Δ

be the fixed part of the linear system $|L| = |C|$. We set $C = A+\Delta$ where A is the moving part. By Lemma 3.2 one sees $\mathrm{Supp}\Delta \cap D = \phi$ and $\Delta \cdot D = 0$. We also have by Lemma 3.2, (2)

$$h^0(\mathcal{O}_Z(C)) \geqq 1+(C^2/2) \geqq 2$$

and thus $A \neq 0$. One may assume that $\mathrm{Supp}A \cap D = \phi$. Note that $A^2 \geqq 0$ since A is the moving part.

Case 1. $A^2 > 0$.

By Proposition 3.3 any general member of $|A|$ is an irreducible curve with arithmetic genus $(A^2/2)+1$ and $h^1(\mathcal{O}_Z(A)) = 1$. One has

$$h^0(\mathcal{O}_Z(A)) = \chi(\mathcal{O}_Z)+(A^2+D\cdot A)/2+h^1(\mathcal{O}_Z(A)) = (A^2/2)+2$$

by the Riemann-Roch formula. On the other hand one has also

$$h^0(\mathcal{O}_Z(A+\Delta)) = ((A+\Delta)^2/2)+2$$

since $h^1(\mathcal{O}_Z(A+\Delta)) = 1$ by Lemma 3.2, (2). It implies that $A^2 = (A+\Delta)^2$ since $h^0(\mathcal{O}_Z(A)) = h^0(\mathcal{O}_Z(A+\Delta))$. We have $2A\cdot\Delta+\Delta^2 = 0$. Now recall that C is numerically effective. Thus

$$0 \leqq C\cdot\Delta = (A+\Delta)\ \Delta = -A\cdot\Delta\ .$$

However $A\cdot\Delta \geqq 0$ since A is the moving part of $|C|$. In conclusion we have $A\cdot\Delta = 0$ and $\Delta^2 = 0$.

If $\Delta \neq 0$, then $\Delta^2 = -2h^1(\mathcal{O}_Z(\Delta)) < 0$ by Lemma 3.5. Therefore $\Delta = 0$, i.e., $|C|$ has no fixed components.

Case 2. $A^2 = 0$.

By Proposition 3.3, there exists a smooth irreducible elliptic curve F and a positive integer k with $\mathcal{O}_Z(A) \cong \mathcal{O}_Z(kF)$ and $F\cdot D$

$= 0$. Let $\Delta_1, \Delta_2, \cdot \quad , \Delta_N$ be connected components of Δ.

We divide the rest of the proof into several lemmas.

Lemma 3.8. For every i, $F\cdot\Delta_i>0$.

Proof. If for some i, $F\cdot\Delta_i = 0$, then by Lemma 3.5

$$0\leq C\cdot\Delta_i = (kF+\Sigma\Delta_j)\ \Delta_i = \Delta_i^{\,2} = -2h^1(\mathcal{O}_Z(\Delta_i))<0,$$

which is a contradiction. Q.E.D.

Let Γ_i be an irreducible component of Δ_i with $F\cdot\Gamma_i>0$.

Lemma 3.9. $k\geq 2$.

Proof. If $k = 1$, then by the same reason as in case 1, we have $A\cdot\Delta = F\cdot\Delta = 0$. However we have just proved that $F\cdot\Delta = \Sigma\, F\ \Delta_i>0$, which is a contradiction. Thus $k\geq 2$. Q.E.D.

Lemma 3.10. $N = 1$.

Proof. Assume $N\geq 2$. Choose general members $F_1, \quad ,F_k\in|F|$ and set $P = F_1+ \quad +F_k+\Gamma_1$. $Q = P+\Gamma_2$. Obviously $\mathrm{Supp}P \cap D = \mathrm{Supp}Q \cap D = \phi$ and $h^0(\mathcal{O}_P) = h^0(\mathcal{O}_Q) = 1$. We have $h^1(\mathcal{O}_Z(P)) = h^1(\mathcal{O}_Z(Q)) = 1$ by Lemma 3.4. By the Riemann-Roch formula we have

$$h^0(\mathcal{O}_Z(P)) = (P^2/2)+2, \qquad h^0(\mathcal{O}_Z(Q)) = (Q^2/2)+2.$$

Since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$ by definition, it implies that

$$P^2 = Q^2 = (P+\Gamma_2)^2 = P^2+2P\cdot\Gamma_2-2.$$

Here note that $\Gamma_2{}^2 = -2$ by Corollary 3.6. We have

$$1 = P\cdot\Gamma_2 = (kF+\Gamma_1)\,\Gamma_2 = kF\,\Gamma_2 \geq k \geq 2,$$

which is a contradiction. Thus $N = 1$. Q.E.D.

Set $\Delta_1 = \Delta = \sum_{j=1}^{J} a_j\Theta_j$ where Θ_j is a mutually different irreducible curve and a_j is a positive integer. We assume that $\Theta_1 = \Gamma_1$. By Corollary 3.6 every Θ_j is a smooth rational curve with $\Theta_j{}^2 = -2$ and $D\cdot\Theta_j = 0$.

Lemma 3.11. $F\cdot\Theta_1 = 1$.

Proof. First note that $h^0(\mathcal{O}_Z(kF)) = 1+k$ by Proposition 3.3 and by the Riemann-Roch formula. Since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(kF))$ for the divisor P in the proof of Lemma 3.9, we have

$$2k+2 = (kF+\Theta_1)^2+4 = 2kF\cdot\Theta_1+2,$$

which implies the lemma. Q.E.D.

Lemma 3.12. $F\,\Theta_i = 0$ if $i \neq 1$.

Proof. Fix an integer i with $i \neq 1$. There exists a subset S of $\{1, 2, \quad , J\}$ with $1\in S$, $i\notin S$, such that $\Delta_S = \sum_{j\in S}\Theta_j$ and $\Delta_S+\Theta_i$ are connected. Set $P = kF+\Delta_S$ and $Q = kF+\Delta_S+\Theta_i$. By the Riemann-Roch formula, we have

$$h^0(\mathcal{O}_Z(P)) = (P^2/2)+2, \qquad h^0(\mathcal{O}_Z(Q)) = (Q^2/2)+2.$$

We have $P^2 = Q^2$ since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$. It implies $(kF+\Delta_S)\cdot\Theta_i = P\cdot\Theta_i = -\Theta_i{}^2/2 = 1$. By the choice of Δ_S, we have $\Delta_S\cdot\Theta_i>0$. Thus $F\cdot\Theta_i = 0$. Q.E.D.

Lemma 3.13. Assume that there is a subset S of $\{1, 2, \quad , J\}$ with $1\in S$ such that $\Delta_S = \sum_{j\in S}\Theta_j$ is connected and $k+\Delta_S\cdot\Theta_1\geqq 2$. Then $a_1 = 1$.

Proof. Set $P = kF+\Delta_S$, $Q = P+\Theta_1$ and $N = \mathcal{O}_Z(Q)|_{\Theta_1}$. Note that $\deg N = (kF+\Delta_S+\Theta_1)\cdot\Theta_1 = k+\Delta_S\cdot\Theta_1-2\geqq 0$ by assumption. One sees easily $h^1(\mathcal{O}_Z(P)) = 1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(P) \longrightarrow \mathcal{O}_Z(Q) \longrightarrow N \longrightarrow 0.$$

We have $h^1(\mathcal{O}_Z(Q))\leqq 1$ since $h^1(N) = 0$. Consider the sequence

$$0 \longrightarrow \mathcal{O}_Z(-Q-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Q\oplus\mathcal{O}_D \longrightarrow 0.$$

It is exact since $\mathrm{Supp}Q \cap D = \phi$. Thus $h^1(\mathcal{O}_Z(Q)) = h^1(\mathcal{O}_Z(-Q-D)) = h^0(\mathcal{O}_Q))\geqq 1$. It follows that $h^1(\mathcal{O}_Z(Q)) = 1$. By Riemann-Roch

$$h^0(\mathcal{O}_Z(P)) = P^2/2+2, \qquad h^0(\mathcal{O}_Z(Q)) = Q^2/2+2.$$

Assume that $a_1\geqq 2$. Then $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$. We have $P^2 = Q^2 = P^2+2P\cdot\Theta_1-2$. Thus $P\cdot\Theta_1 = 1$.

On the other hand by definition of P and by assumption $P\cdot\Theta_1 = (kF+\Delta_S)\cdot\Theta_1 = k+\Delta_S\ \Theta_1\geqq 2$. We get a contradiction. Q.E.D.

Lemma 3.14. If $a_1 = 1$, then $F\cdot\Delta = 1$ and $\Delta^2 = -2$.

Proof. Assume $a_1 = 1$. We write $\Delta = \Theta_1+\Delta'$. Since $\Delta'\cdot F = 0$ by

Lemma 3.12, we have $F\cdot\Delta = F\cdot\Theta_1 = 1$. By Riemann-Roch we have $h^0(\mathcal{O}_Z(kF)) = 1+k$ and $h^0(\mathcal{O}_Z(kF+\Delta)) = (kF+\Delta)^2/2+2$. Since these two numbers are equal, we have $(kF+\Delta)^2 = 2k-2$. It implies $\Delta^2 = -2$ since $F^2 = 0$ and $F\cdot\Delta = 1$. Q.E.D.

Lemma 3.15. If $k\geq 4$, then $\Delta = \Theta_1$.

Proof. We assume $k\geq 4$. Set $S = \{1\}$. The assumption of Lemma 3.13 is satisfied. Thus we have $a_1 = 1$ and $\Delta^2 = -2$ by Lemma 3.13 and Lemma 3.14. Set $\Delta' = \Delta-\Theta_1$. The divisor Δ' does not contain Θ_1. Assume $\Delta' \neq 0$. Then $\Delta'\cdot\Theta_1>0$ since Δ is connected. It follows from the equality

$$-2 = \Delta^2 = (\Theta_1+\Delta')^2 = -2+\Delta'\cdot\Theta_1+\Delta\cdot\Delta'$$

that $\Delta\cdot\Delta'<0$. However, since C is numerically effective and $F\cdot\Delta' = 0$ by Lemma 3.12, we have that $0\leq C\cdot\Delta' = (kF+\Delta)\,\Delta' = \Delta\cdot\Delta'$, a contradiction. Thus $\Delta' = 0$. Q.E.D.

Lemma 3.16. If $k = 3$, then $\Delta = \Theta_1$.

Proof. We assume $k = 3$. Moreover assume $\Delta' = \Delta-a_1\Theta_1 \neq 0$. There exists a suffix i with $\Theta_i\cdot\Theta_1 \neq 0$. Set $S = \{1, i\}$. Since $k+\Delta_S\,\Theta_1 = 3+\Theta_i\,\Theta_1-2$, the assumption of Lemma 3.13 is satisfied. Thus we have $a_1 = 1$ and $\Delta^2 = -2$. By the same reasoning as in Lemma 3.15, one obtains a contradiction. Thus $\Delta = a_1\Theta_1$.

By the same reasoning as in Lemma 3.14 one sees $4 = (3F+\Delta)^2 =$

$(3F+a_1\Theta_1)^2 = 6a_1-2a_1^2$ since $F\cdot\Theta_1 = 1$ and $\Theta_1^2 = -2$. We have $a_1 = 1$ or 2. If $a_1 = 2$, then $C\cdot\Theta_1 = (3F+2\Theta_1)\cdot\Theta_1 = -1$, that is, C is not numerically effective. We have consequently $\Delta = \Theta_1$.

Q.E.D.

Lemma 3.17. If $k = 2$, then $\Delta = \Theta_1$.

Proof. We assume that $k = 2$. Moreover assume that $a_1 = 1$. Set $\Delta' = \Delta-\Theta_1$. We have $\Delta'\cdot\Theta_1 \geqq 0$ since Δ' does not contain Θ_1. By Lemma 3.12 we have also $\Delta\cdot\Delta' = (2F+\Delta)\,\Delta' = C\cdot\Delta' \geqq 0$. On the other hand by Lemma 3.5 $\Delta^2 \leqq -2$. We have

$$-2 \geqq \Delta^2 = (\Theta_1+\Delta')^2 = -2+\Delta'\cdot\Theta_1+\Delta\cdot\Delta' \geqq -2.$$

It implies that $\Delta'\cdot\Theta_1 = \Delta\cdot\Delta' = 0$, $\Delta^2 = -2$. We have $\Delta'^2 = \Delta\cdot\Delta'-\Theta_1\cdot\Delta' = 0$. But $\Delta'^2<0$ if $\Delta' \neq 0$ by Lemma 3.5. Thus $\Delta' = 0$.

Next assume that $a_1 \geqq 2$. Since $0 \leqq L\cdot\Theta_1 = (2F+a_1\Theta_1)\cdot\Theta_1+\sum_{i\neq 1} a_i\Theta_i\cdot\Theta_1 = 2-2a_1+\sum_{i\neq 1} a_i\Theta_i\cdot\Theta_1$ there is an index i with $i \neq 1$, $\Theta_i\cdot\Theta_1>0$. If there are two indices i, j, $1 \neq i \neq j \neq 1$ with $\Theta_i\cdot\Theta_1>0$, $\Theta_j\cdot\Theta_1>0$, setting $S = \{1, i, j\}$ we have $a_1 = 1$ by Lemma 3.13. Thus for some unique index i_2 $\Theta_{i_2}\cdot\Theta_1>0$. By renumbering if necessary we can assume $i_2 = 2$. We have that $\Theta_2\,\Theta_1 = 1$ since $0>(\Theta_1+\Theta_2)^2 = -4+2\Theta_1\cdot\Theta_2$ by Lemma 3.5. We have the next inequality.

$$a_2-2a_1+2 = L\cdot\Theta_1 \geqq 0 \tag{3.1}$$

In particular $a_2 \geqq 2$. Now since $0 \leqq L\cdot\Theta_2 = a_1-2a_2+\sum_{i>2} a_i\Theta_i\cdot\Theta_2$, there is an index $i>2$ with $\Theta_i\,\Theta_2>1$. Assume that for mutually different three indices $i_\alpha > 2$, $\alpha = 1, 2, 3$, $\Theta_{i_\alpha}\cdot\Theta_2 > 0$ holds. Set $P_1 = 2F + \Theta_1 + \Theta_2 + \sum_{\alpha=1}^{3}\Theta_{i_\alpha}$ and $Q_1 = P_1 + \Theta_2$. Since $\mathcal{O}_Z(Q_1)|_{\Theta_2} \cong \mathcal{O}_{\Theta_2}$ and $\Theta_2 \cong \mathbb{P}^1$ and since $h^0(\mathcal{O}_Z(P_1)) = h^0(\mathcal{O}_Z(Q_1))$ it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(P_1) \longrightarrow \mathcal{O}_Z(Q_1) \longrightarrow \mathcal{O}_{\Theta_2} \longrightarrow 0$$

that $h^1(\mathcal{O}_Z(Q_1)) = 0$. However by the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-Q_1-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Q_1} \oplus \mathcal{O}_D \longrightarrow 0$$

we have $h^1(\mathcal{O}_Z(Q_1)) = h^1(\mathcal{O}_Z(-Q_1-D)) \geq 1$, a contradiction. Thus renumbering if necessary we can assume that one of the following two assertions holds for $k = 3$.

$(1)_k$ $\Theta_k\cdot\Theta_{k-1} = 1$ and $\Theta_i\cdot\Theta_{k-1} = 0$ for $i > k$.

$(2)_k$ $\Theta_k\cdot\Theta_{k-1} = \Theta_{k+1}\cdot\Theta_{k-1} = 1$ and $\Theta_i\cdot\Theta_{k-1} = 0$ for $i > k+1$.

For a moment assume that case $(1)_3$ takes place. Since

<3.2> $\qquad L\cdot\Theta_2 = a_1 - 2a_2 + a_3 \geq 0$

and by <3.1>, we have $a_3 \geq 2$. Repeating the similar argument as just the above one sees that we can assume that $(1)_4$ or $(2)_4$ holds. If $(1)_4$ takes place, inequalities

<3.k> $\qquad L\cdot\Theta_k = a_{k-1} - 2a_k + a_{k+1} \geq 0$

$k = 2, 3$ and <3.1> implies that $a_4 \geq 2$ and we can repeat the similar discussion more. Since inequalities <3.k> $1 \leq k \leq K$ implies $a_{K+1} \geq 2$ and since the number of irreducible components of Δ is finite, we can consequently assume that $(2)_K$ takes place for some $K \geq 2$. Set $\Sigma = \Theta_1 + \Theta_2 + \quad + \Theta_K$, $P_2 = 2F + \Sigma + \Theta_{K+1} + \Theta_{K+2}$, and $Q_2 = P_2 + \Sigma$. We can see easily that $\mathcal{O}_Z(Q_2)|_\Sigma \cong \mathcal{O}_\Sigma$, $h^0(\mathcal{O}_\Sigma) = 1$ and $h^1(\mathcal{O}_\Sigma) = 0$. Now

$h^1(\mathcal{O}_Z(P_2)) = 1$ by Lemma 3.4. and $h^0(\mathcal{O}_Z(P_2)) = h^0(\mathcal{O}_Z(Q_2))$ since Δ is a sum of $2\Sigma+\Theta_{K+1}+\Theta_{K+2}$ and some effective divisor. It follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(P_2) \longrightarrow \mathcal{O}_Z(Q_2) \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0$$

that $h^1(\mathcal{O}_Z(Q_2)) = 0$. On the other hand since the sequence

$$0 \longrightarrow \mathcal{O}_Z(-Q_2-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Q_2} \oplus \mathcal{O}_D \longrightarrow 0$$

is exact, we have $h^1(\mathcal{O}_Z(Q_2)) = h^1(\mathcal{O}_Z(-Q_2-D)) = h^0(\mathcal{O}_{Q_2}) \geqq 1$, a contradiction. Thus the case $a_1 \geqq 2$ never takes place. Q.E.D.

The above lemma completes the proof of Proposition 3.7.

Proposition 3.18. Let C be an effective divisor on Z with $C \cdot D = 0$. Assume that the linear system $|C|$ has no fixed components and that $C^2 = 2$ or 4. Then $|C|$ has no fixed points.

Proof. Assume that $|C|$ has no fixed components but it has isolated fixed points.

By induction we define a sequence of blowing-ups,

$$\hat{Z} = Z_{(k)} \xrightarrow{\tau_k} Z_{(k-1)} \xrightarrow{\tau_{k-1}} \qquad \longrightarrow Z_{(2)} \xrightarrow{\tau_2} Z_{(1)} \xrightarrow{\tau_1} Z_{(0)} = Z$$

an integer m_j for $1 \leqq j \leqq k$ and a line bundle L_j on $Z_{(j)}$ for $0 \leqq j \leqq k$ as follows. First of all set $Z_{(0)} = Z$ and $L_0 = \mathcal{O}_Z(C)$. Next assume that $Z_{(i)}$, τ_i, m_i, L_i have been constructed for $0 \leqq i \leqq j-1$. If $|L_{j-1}|$ has no fixed points, then setting $k = j-1$ and $\hat{Z} = Z_{(j-1)}$, we terminate the procedure. If $|L_{j-1}|$ has fixed points, then let $\tau_j\colon Z_{(j)} \longrightarrow Z_{(j-1)}$ be the blowing-up of one of the fixed points $z_j \in Z_{(j-1)}$. Set $m_j = \min\{ \mathrm{mult}_{z_j}(A) \mid A \in |L_{j-1}| \}$, where $\mathrm{mult}_z(A)$ denotes the multiplicity of the curve A at z. We define $L_j = (\tau_j^* L_{j-1}) \otimes \mathcal{O}_{Z_{(j)}}(-m_j \tau^{-1}(z_j))$. We have $L_j^2 \geq 0$ for every j since $|L_j| \neq \phi$ and $|L_j|$ has no fixed components. Since $L_j^2 = L_{j-1}^2 - m_j^2 < L_{j-1}^2$ this procedure terminates in finite steps.

Set $\hat{L} = L_k$. If $\hat{L}^2 = 0$, then the image of the rational map $\phi_L : Z \dashrightarrow \mathbb{P}^N$ associated to the line bundle $L = \mathcal{O}_Z(C)$ has dimension ≤ 1. We have $L^2 = C^2 = 0$ by Proposition 3.3, which contradicts to the assumption. Thus $\hat{L}^2 > 0$.

Next we show that $p_a(A) \leq 1$ for any general member A of $|\hat{L}|$.

Case 1. $C^2 = 2$.

Note that $h^1(L) = 1$ by Proposition 3.3. We have $h^0(\hat{L}) = h^0(L) = C^2/2+2 = 3$ by Riemann-Roch. We have a morphism $\phi_{\hat{L}} : \hat{Z} \longrightarrow \mathbb{P}^2$. On the other hand $\hat{L}^2 = 1$ since $0 < \hat{L} < 2 = C^2$. Thus any general member A of $|\hat{L}|$ has a morphism of degree 1 to a line in $\mathbb{P}^2$. Thus $p_a(A) = 0$.

Case 2. $C^2 = 4$.

We have a morphism $\phi_{\hat{L}} : \hat{Z} \longrightarrow \mathbb{P}^3$ since $h^0(\hat{L}) = h^0(L) = 4$ by Riemann-Roch. Since $0 < \hat{L}^2 < L^2 = C^2 = 4$, one sees that $\phi_{\hat{L}}$ is a generically one to one morphism whose image is an irreducible cubic surface or an irreducible quadratic surface. Then any general member A of $|\hat{L}|$ has a morphism of degree 1 to either a plane irreducible cubic curve or a plane irreducible quadratic curve. Thus

$p_a(A) \leqq 1$.

We know $p_a(A) \leqq 1$ in any case.

Now let $E_1, \cdots, E_k$ be the total inverse image on $\hat{Z}$ of the curve $\tau_1^{-1}(z_1), \cdots, \tau_k^{-1}(z_k)$. We have

$\hat{L} = (\tau^* L)(-m_1E_1 - m_2E_2 - \cdots - m_kE_k)$, $\omega_{\hat{Z}} = (\tau^* \omega_Z)(E_1 + E_2 + \cdots + E_k)$

where $\tau = \tau_1\tau_2 \cdots \tau_k$. Thus we have $\hat{L} \cdot \omega_{\hat{Z}} = C \cdot \omega_Z + \Sigma m_i = \Sigma m_i$. By the adjunction formula

$$p_a(A) = (\hat{L}^2 + \omega_{\hat{Z}} \cdot \hat{L})/2 + 1 = (\hat{L}^2/2) + (\Sigma m_i/2) + 1 \geqq 2.$$

We obtain a contradiction. Thus $|C|$ has no fixed points. Q.E.D.

Lemma 3.19. Let L be a polarization on Z.

(1) If an irreducible curve A on Z satisfies $L \cdot A = 0$, then either A coincides with D or it is a smooth rational curve with $A^2 = -2$ and $A \cap D = \phi$.

(2) Let $\underline{E}$ be the union of irreducible curves A with $L \cdot A = 0$ and $\underline{E}_0$ be a connected component of $\underline{E}$. Let $A_1, A_2, \cdots, A_k$ be all the irreducible curves contained in $\underline{E}_0$. Then the intersection matrix $(A_i \cdot A_j)_{1 \leqq i,j \leqq k}$ is negative definite.

(3) Unless $\underline{E}_0 = D$, $\underline{E}_0$ is the support of the exceptional curves in the minimal resolution of a rational double point.

Proof. We can assume that $A \neq D$. Under this assumption we have $A \cdot D \geqq 0$. By the Hodge index theorem we have also $A^2 < 0$. By the adjunction formula $0 \leqq p_a(A) = (A^2 - A \cdot D)/2 + 1$. We have either $A^2 = -1$ and $A \cdot D = 1$ or $A^2 = -2$ and $A \cdot D = 0$. In any case $p_a(A) = 0$.

It is well-known that if $p_a(A) = 0$, then A is a smooth rational curve. If $A^2 = -1$ and $A \cdot D = 1$, then A is an exceptional curve of the first kind. Since L is a polarization we have $A \cdot L > 0$, which contradicts to the choice of A. Thus $A^2 = -2$ and $A \cdot D = 0$. The last equality implies $A \cap D = \phi$. (2) is an easy consequence of the Hodge index theorem. (3) follows from (1) and (2). (Cf. Artin [2]) Q.E.D.

By the well-known Grauert's theorem, (Cf. Grauert [7]) we can contract all the connected components of $\underline{E}$ to isolated normal singular points. Let $\rho: Z \longrightarrow X$ be the contraction morphism. Here X is a normal surface with a unique singular point with positive geometric genus at $w = \rho(D)$ and sevaral rational double points.

Proposition 3.20. Assume that a polarization L on Z defines a morphism $\phi = \phi_L: Z \longrightarrow \mathbb{P}^N$. Then we have a finite morphism $\bar{\Phi}$: $X \longrightarrow \mathbb{P}^N$ with $\phi = \bar{\Phi} \circ \rho$.

$$\begin{tikzcd} Z \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & \mathbb{P}^N \\ & X \arrow[ur, "\bar{\Phi}"'] & \end{tikzcd}$$

Proof. Set $\rho(\underline{E}) = S$. Note that $\rho | Z-\underline{E}$: $Z-\underline{E} \longrightarrow X-S$ is an isomorphism. Thus we can define a morphism $\bar{\Phi} = \phi \circ (\rho | Z-\underline{E})^{-1}$. Since $\phi(\underline{E})$ is a set of isolated points and X is normal, we can extend $\bar{\Phi}$ to whole X. Obviously the resulting morphism $X \longrightarrow \mathbb{P}^N$ is proper. Assume that there exists a point $z \in \mathbb{P}^N$ such that $\bar{\Phi}^{-1}(z)$ has dimension 1. Let A be an irreducible curve contained in $\bar{\Phi}^{-1}(z)$. Let $\hat{A}$ be the strict inverse image of A by ρ. We have $L \cdot \hat{A} = 0$. Thus $\hat{A} \subset \underline{E}$ and $\rho(\hat{A}) = A$ is a point, which is a contradiction. Thus $\bar{\Phi}$ is a finite morphism. Q.E.D.

Proposition 3.21. Assume that a polarization L on Z defines a morphism $\phi = \phi_L : Z \longrightarrow \mathbb{P}^3$ of degree 2 whose image is a quadratic surface. We have a smooth irreducible elliptic curve F on Z with $L \cdot F = 2$, $F \cap D = \phi$ and $F^2 = 0$.

Proof. <u>Case A</u>. Assume the image of ϕ is a smooth quadratic surface Σ. Let $p: \Sigma \longrightarrow \mathbb{P}^1$ be the composition of an isomorphism $\Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and the projection to a factor $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Choose a general point $z \in \mathbb{P}^1$ and set $G = p^*(z)$ and $F = \phi^*(G)$. F is irreducible. We have $F \cap D = \phi$ since $\phi(D)$ and $p\phi(D)$ are isolated points by assumption $L|_D \cong \mathcal{O}_D$. We have $L \cdot F = 2\mathcal{O}_{\mathbb{P}^3}(1) \cdot G = 2$ and $F^2 = 2G^2 = 0$. Obviously the linear system $|F|$ has no fixed compoments. By Proposition 3.3, one sees that F is a smooth elliptic curve.

<u>Case B</u>. Assume that the image of ϕ is a quadratic surface Σ_0 with a unique singular point $v \in \Sigma_0$.

Lemma 3.22. If $\phi(D) = \{v\}$, then $\phi^{-1}(v) = D$.

Proof. Set $\{w\} = \rho(D)$, $w \in X$. Note that $\bar{\Phi}(w) = \{v\}$ by assumption. Let U be a sufficiently small neighbourhood of $v \in U \subset \Sigma_0$. Let V be the connected component of $\bar{\Phi}^{-1}(U)$ containg w. Let $S \subset V-\{w\}$ be the discriminant of $\bar{\Phi}|V-\{w\}$.

Case 1. Assume that the closure of $\bar{\Phi}(S)$ in U does not contain v. By choosing a smaller U, we can assume that $\bar{\Phi}|V-\{w\}$ is unramified. Note that $\pi_1(U-\{v\}) \cong \mathbb{Z}/2\mathbb{Z}$ since the A_1-singularity (U,v) is the quotient of $(\mathbb{C}^2,0)$ by the action of $\mathbb{Z}/2\mathbb{Z}$ defined by $(x, y) \longrightarrow (-x, -y)$. Thus $\pi_1(V-\{w\})$ is either a trivial group $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$. If $\pi_1(V-\{w\}) = \{e\}$, then $w \in X$ is a simple point by a Mumford's theorem. (Cf. Mumford [14]) If it is $\mathbb{Z}/2\mathbb{Z}$, $\bar{\Phi}|V-\{w\}$ is an isomorphism. Since V and U are normal, it induces an isomorphism $\bar{\Phi}|V\colon V \longrightarrow U$. Thus $w \in X$ is a A_1-singular point. However by the construction we have $p_g(X, w) \geq 1$. Therefore one sees that our Case 1 never takes place under our assumption.

Case 2. Next we assume that the closure of $\bar{\Phi}(S)$ in U contains v. Since $\bar{\Phi}$ is a finite morphism of degree 2, the set $\{ x \in U \mid \#\bar{\Phi}^{-1}(x) = 1 \}$ coincides with the closure of $\bar{\Phi}(S)$ in U. Thus $\#\bar{\Phi}^{-1}(v) = 1$. We have $\{w\} = \bar{\Phi}^{-1}(v)$. It implies $\phi^{-1}(v) = \rho^{-1}(w)$. Under the assumption of the lemma $D \subset \phi^{-1}(v)$. However since $\rho^{-1}\rho(D) = D$ by the definition of ρ, we have $\phi^{-1}(v) = D$. Q.E.D.

Lemma 3.23. Let G be a general member of the ruling $\mathbb{P}^1$-family of Σ_0 and F be the strict inverse image of G by ϕ. We have $\dim |F| = 1$ and $|F|$ has no fixed components.

Proof. We define a linear system Λ on Z by $\Lambda = \{ \phi^*P \mid P$ is a plane in $\mathbb{P}^3$ with $v \in P \}$. Let Δ be the fixed components of Λ. Obviously we have $\mathrm{Supp}\,\Delta \subset \phi^{-1}(v)$. Let P_0 be a general plane in $\mathbb{P}^3$ passing through v. We set $P_0 \cap \Sigma_0 = G \cup G'$ where G and G' are members of the ruling $\mathbb{P}^1$-family of Σ_0. Let F (resp. F') be the strict inverse image of G (resp. G') by ϕ. We have $F+F'+\Delta \in \Lambda$.

Moreover we define a 1-dimensional linear system Ξ by

$$\Xi = \{ \phi^*P-F'-\Delta \mid P \text{ is a plane in } \mathbb{P}^3 \text{ with } P \supset G' \}.$$

We have $|F| \supset \Xi$ since $F \in \Xi$. Let $A \in |F|$ be an arbitrary member. $A+F'+\Delta \in |L|$ since $F+F'+\Delta \in |L|$. Thus there is a plane P_1 in $\mathbb{P}^3$ with $A+F'+\Delta = \phi^*P_1$ because $|L|$ is a complete linear system. P_1 necessarily contains G'. It implies that $A \in \Xi$. Thus $|F| = \Xi$, which concludes the proof. Q.E.D.

Lemma 3.24. $\phi(D) \neq \{v\}$.

Proof. Assume that $\phi(D) = \{v\}$. We will deduce a contradiction.

Let F be a divisor as in Lemma 3.23. By the Riemann-Roch formula and by Lemma 3.23, we have

$$2 = 1+\dim |F| = (F^2+F\cdot D)/2+1+h^1(\mathcal{O}_Z(F)).$$

Lemma 3.23 also implies $F^2 \geqq 0$. Since $\phi^{-1}(v) = D$ by Lemma 3.22, we have $F\cdot D>0$. One sees that only one of the following two choices takes place.

(a) $F^2 = 0$, $F\cdot D = 2$ and $h^1(\mathcal{O}_Z(F)) = 0$

(b) $F^2 = 1$, $F\cdot D = 1$ and $h^1(\mathcal{O}_Z(F)) = 0$

Now there exist integers m_1, m_2 such that $F' = F+m_1D$, $\Delta = m_2D$ where F' and Δ are divisors defined in the proof of Lemma 3.23. Therefore $|L|\ni F+F'+\Delta\sim 2F+mD$ with $m = m_1+m_2$.

We consider case (a). We have $4 = L^2 = (2F+mD) = 8m-m^2$. However the quadratic equation $m^2-8m+4 = 0$ has no integral solution, which is a contradiction.

Next we consider case (b). We have $4 = (2F+mD)^2 = 4+4m-m^2$. Thus $m = 0$ or 4. In both cases we have a line bundle M with $L = 2M$ in $\mathrm{Pic}(Z)$. Since M belongs to the orthogonal complement Q of $\mathbb{Z}\omega_Z$ and since Q is an even lattice $4 = L^2 = 4M^2$ is a multiple of 8, which is a contradiction. Thus $\phi(D) \neq \{v\}$. Q.E.D.

Now we go back to the proof of Proposition 3.21, Case B. By Lemma 3.24, we can choose a general member G of the ruling $\mathbb{P}^1$-family of Σ_0 with $G \cap \phi(D) = \phi$. Let F be the strict inverse image of G by ϕ. We have $\mathcal{O}_Z(F)|_D \cong \mathcal{O}_D$. By Riemann-Roch $2 = (F^2/2)+1+h^1(\mathcal{O}_Z(F))$. One sees that only one of the following two choices takes place.

(c) $F^2 = 2$ and $h^1(\mathcal{O}_Z(F)) = 0$

(d) $F^2 = 0$ and $h^1(\mathcal{O}_Z(F)) = 1$.

Now note that $h^2(\mathcal{O}_Z(F-D)) = h^0(\mathcal{O}_Z(-F)) = 0$ by the Serre duality. It implies that the map $H^1(\mathcal{O}_Z(F)) \longrightarrow H^1(\mathcal{O}_Z(F)|_D) \cong$

$H^1(\mathcal{O}_D) \cong \mathbb{C}$ is surjective. Thus $h^1(\mathcal{O}_Z(F)) \geq 1$ and case (c) never takes place. The equality $L \cdot F = 2$ is obvious by definition. It concludes the proof of Proposition 3.21. Q.E.D.

Theorem 3.25. Let L be a polarization of degree 4 on a rational surface Z with an irreducible effective anti-canonical divisor D. The following conditions are equivalent.

(1) The rational map ϕ_L associated to L defines a birational morphism to a quartic surface in $\mathbb{P}^3$.

(2) There exists no element $M \in \mathrm{Pic}(Z)$ with $M^2 = 0$, $M \cdot L = 2$ and $M|_D \cong \mathcal{O}_D$.

Besides if one of the above equivalent conditions holds, then the induced morphism $\bar{\phi}: X \longrightarrow \mathbb{P}^3$ by ϕ_L is an embedding.

Proof. First we show (2)$\Rightarrow$(1). Assume that $|L|$ has fixed components. By Proposition 3.7 there exists a smooth irreducible elliptic curve F and a smooth irreducible rational curve Γ with $F^2 = 0$, $F \cdot D = 0$. $\Gamma^2 = -2$, $\Gamma \cdot D = 0$, $\Gamma \cdot F = 1$ and $L \cong \mathcal{O}_Z(3F+\Gamma)$. The line bundle $M = \mathcal{O}_Z(F+\Gamma)$ satisfies the conditions in (2). Next assume that $|L|$ has no fixed components. By Proposition 3.18 $|L|$ has no fixed points. Thus ϕ_L is a morphism. By Lemma 3.2 one sees ϕ_L maps Z to $\mathbb{P}^3$. Since $L^2 = 4$, the image of ϕ_L is either a quadratic surface or a quartic surface. Assume moreover that $\mathrm{Im}\,\phi_L$ is a quadratic surface. By Proposition 3.21 we have a smooth elliptic curve F on Z with $F^2 = 0$, $F \cdot D = 0$ and $L\ F =$

2. The line bundle $M = \mathcal{O}_Z(F)$ satisfies (2). Thus (2) implies (1).

Next we show (1)⟹(2). Assume that there is an element $M \in \mathrm{Pic}(Z)$ with $M^2 = 0$, $M \cdot L = 2$, $M|_D \cong \mathcal{O}_D$ and that ϕ_L is a birational morphism to a quartic surface in $\mathbb{P}^3$. We will deduce a contradiction. By Riemann-Roch we have $h^0(M)+h^2(M) \geqq 1$. If $h^2(M) = h^0(-M+\omega_Z) \neq 0$, we have $(-M+\omega_Z)\cdot L \geqq 0$ since L is numerically effective. However we have $(-M+\omega_Z)\cdot L = -2+0 = -2$, a contradiction. Thus $h^2(M) = 0$ and $h^0(M) \neq 0$, i.e., M is effective. Let A be an effective divisor with $M \cong \mathcal{O}_Z(A)$. We set

$$A = mD + \sum_{i=1}^{k} n_i A_i + F$$

where k, m, n_1, , n_k are integers with $k \geqq 0$, $m \geqq 0$, $n_i \geqq 1$ $(1 \leqq i \leqq k)$, A_1, , A_k are mutually different irreducible curves with $A_i \neq D$, $A_i \cdot D > 0$ for every i and F is an effective divisor with $\mathrm{Supp}\, F \cap D = \phi$. Let $\underline{E}$ be the union of exceptional curves of $\rho\colon Z \longrightarrow X$. Since D is a connected component of $\underline{E}$ and since $A_i \cdot D > 0$, $\rho(A_i)$ has dimension 1 for every i. Thus $L \cdot A_i > 0$ for every i. Since

$$2 = M \cdot L = mD \cdot L + \sum_{i=1}^{k} n_i A_i\ L + F \cdot L = \sum_{i=1}^{k} n_i A_i \cdot L + F \cdot L$$

we have 4 cases.

<1> $k = 0$.

<2> $k \geqq 1$ and $n_i = 1$ for every i.

<3> $k = 1$, $n_1 = 2$, $m \geqq 1$, $A_1^2 \geqq 0$.

<4> $k = 1$, $n_1 = 2$, $m \geqq 1$, $A_1^2 < 0$.

(Note that $k = 0$ if and only if $m = 0$.)

Now we need two lemmas.

Lemma 3.26. Consider a divisor $A = mD+\sum_{i=1}^{k} A_i+F$ satisfying the following conditions.

(i) $k\geq 1$, $m\geq 1$

(ii) $D\cdot A_i>0$, $A_1, \cdot \quad , A_k$ are mutually different irreducible divisors.

(iii) Supp $F \cap D = \phi$ and F is an effective divisor.

(iv) $\mathcal{O}_Z(A)|_D \cong \mathcal{O}_D$.

Then A is linearly equivalent to a divisor containing no D.

Proof. By induction we show that $H^1(\mathcal{O}_Z(\sum_{i=1}^{j} A_i)) = 0$. If $j = 0$, it is trivial. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(\sum_{i=1}^{j} A_i) \longrightarrow \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i) \longrightarrow \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}} \longrightarrow 0$$

Since $\deg \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}} = A_{j+1}{}^2+\sum_{i=1}^{j} A_i\cdot A_{j+1}>A_{j+1}{}^2-D\cdot A_{j+1}$

$\geq 2p_a(A_{j+1})-2$, we have $H^1(\mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}}) = 0$. By the above sequence and by induction hypothesis we have $H^1(\mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)) = 0$.

Next by induction we show that $H^1(\mathcal{O}_Z(nD+\sum_{i=1}^{k} A_i)) = 0$ for $0\leq n<m$. We have just shown it when $n = 0$. Assume $n\leq m-2$. Set $N = \mathcal{O}_Z((n+1)D+\sum_{i=1}^{k} A_i)|_D$. Since $\deg N = \deg \mathcal{O}_Z(-(m-n-1)D)|_D = -(m-n-1)D^2>0$. We have $H^1(N) = 0$. By the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Z(nD+\sum_{i=1}^{k} A_i) \longrightarrow \mathcal{O}_Z((n+1)D+\sum_{i=1}^{k} A_i) \longrightarrow N \longrightarrow 0$$

we have inductively $H^1(\mathcal{O}_Z((n+1)D+\sum_{i=1}^{k} A_i)) = 0$.

Note that in particular $H^1(\mathcal{O}_Z((m-1)D+\sum_{i=1}^{k} A_i)) = 0$. It implies that $H^0(\mathcal{O}_Z(A')\longrightarrow H^0(\mathcal{O}_Z(A')|_D) \cong H^0(\mathcal{O}_D) = \mathbb{C}$ is surjective where $A' = mD+\sum_{i=1}^{k} A_i$. Surjectivity implies that there exists a divisor A'' linearly equivalent to A' which contains no D. Since $\tilde{A}\sim\tilde{A}''+F$, we have the desired result. Q.E.D.

Lemma 3.27. Let A be an effective divisor with $\mathcal{O}_Z(A)|_D \cong \mathcal{O}_D$ and with $A^2 \geqq 0$. We have $h^0(\mathcal{O}_Z(A))\geqq 2$.

Proof. Note that $h^2(\mathcal{O}_Z(A-D)) = h^0(\mathcal{O}_Z(-A)) = 0$. It implies that $H^1(\mathcal{O}_Z(A))\longrightarrow H^1(\mathcal{O}_Z(A)|_D) \cong H^1(\mathcal{O}_D) \cong \mathbb{C}$ is surjective. Thus $h^1(\mathcal{O}_Z(A))\geqq 1$. By Riemann-Roch, we have

$$h^0(\mathcal{O}_Z(A)) = (A^2+A\cdot D)/2+1+h^1(\mathcal{O}_Z(A))\geqq 2. \qquad \text{Q.E.D.}$$

We continue the proof of Theorem 3.25.

<u>Case <1>.</u> In this case $\operatorname{Supp} A \cap D = \phi$. Let Δ be the fixed components of the linear system $|A|$. Set $C = A-\Delta$. By Lemma 3.27, we have $C \neq 0$ and $C^2\geqq 0$. We first consider the case $C^2 = 0$. By Proposition 3.3 we have a smooth irreducible elliptic curve G with $G^2 = 0$, $G \cap D = \phi$ and an positive integer p with $C\in|pG|$. We have $\Delta\cdot L\geqq 0$ and $G\cdot L\geqq 0$ since L is numerically effective. Since the condition $G\cdot L = 0$ implies $G^2<0$ by the Hodge index theorem, we have moreover $G\cdot L>0$. Now since $2 = A\cdot L = pG\cdot L+\Delta\cdot L$, one sees

that $G\cdot L = 1$ or 2. Secondly we consider the case $C^2>0$. By Proposition 3.3 we can assume that C is an irreducible curve with $p_a(C) = (C^2/2)+1$. Since the condition $C\cdot L = 0$ implies $C^2<0$, we have $C\cdot L>0$. Thus it follows from the equality $C\cdot L+\Delta\cdot L = 2$ that $C\cdot L = 1$ or 2.

Anyway one sees that there exists an irreducible curve C_1 on Z with $p_a(C_1)\geqq 1$, $C_1 \cap D = \phi$ and $C_1\cdot L = 1$ or 2. Since $\phi\colon Z \longrightarrow \mathbb{P}^3$ is generically one-to-one, and since $\dim |C_1|\geqq 1$, we can assume that $\phi|_{C_1}\colon C_1 \longrightarrow \mathbb{P}^3$ is a birational morphism. The image of $\phi|_{C_1}$ is a line or a curve of degree 2 in $\mathbb{P}^3$ since $C_1\cdot L = 1$ or 2. Because such curves have arithmetic genus 0, we have $p_a(C_1)\leqq 0$, a contradiction.

<u>Case <2>.</u> This case is reduced to Case <1> by Lemma 3.26.

<u>Case <3>.</u> First we show $H^1(\mathcal{O}_Z(\ell A_1)) = 0$ for $\ell = 0, 1, 2$ by induction. Since Z is rational, the case $\ell = 0$ is trivial. Assume $\ell\geqq 0$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(\ell A_1) \longrightarrow \mathcal{O}_Z((\ell+1)A_1) \longrightarrow \mathcal{O}_Z((\ell+1)A_1)|_{A_1} \longrightarrow 0.$$

We have $H^1(\mathcal{O}_Z((\ell+1)A_1)|_{A_1}) = 0$ because $\deg \mathcal{O}_Z((\ell+1)A_1)|_{A_1} = (\ell+1)A_1{}^2\geqq A_1{}^2>A_1{}^2-A_1\cdot D = 2p_a(A_1)-2$. By induction hypothesis we have $H^1(\mathcal{O}_Z((\ell+1)A_1)) = 0$. Secondly we show $H^1(\mathcal{O}_Z(nD+2A_1)) = 0$ for $0\leqq n<m$ by induction as well. The case $n = 0$ has been verified. Assume $0\leqq n<m-1$ and consider the sequence

$$0 \longrightarrow \mathcal{O}_Z(nD+2A_1) \longrightarrow \mathcal{O}_Z((n+1)D+2A_1) \longrightarrow \mathcal{O}_Z((n+1)D+2A_1)|_D \longrightarrow 0.$$

Note that $D^2 = \omega_Z{}^2 = 9-t<0$ by Lemma 3.2, (1) and that $\mathcal{O}_Z(A)|_D \cong \mathcal{O}_D$. Thus we have $\deg \mathcal{O}_Z((n+1)D+2A_1)|_D =$

deg $\mathcal{O}_Z(-(m-n-1)D)|_D = -(m-n-1)D^2>0$ and $H^1(\mathcal{O}_Z((n+1)D+2A_1)|_D) = 0$. By the last equality and by the induction hypothesis, we have $H^1(\mathcal{O}_Z((n+1)D+2A_1)) = 0$.

Now in particular $H^1(\mathcal{O}_Z((m-1)D+2A_1)) = 0$. This implies that $H^0(\mathcal{O}_Z(mD+2A_1)) \longrightarrow H^0(\mathcal{O}_Z(mD+2A_1)|_D) \cong H^0(\mathcal{O}_D) \cong \mathbb{C}$ is surjective. Thus there exists a member $A' \in |mD+2A_1|$ which contains no D. We have a divisor $A'+F \in |A|$ containing no D.

<u>Case <4>.</u> This is the last case. Since $A_1{}^2<0$ and $A_1\cdot D>0$, A_1 is an exceptional curve of the first kind. Since there are on Z at most countably many divisors with the form $mD+2E$ where E is an exceptional curve of the first kind, if $mD+2A_1$ is not contained in the fixed components of $|A|$, then there is a divisor $A' \in |A|$ with the form in cases <1>, <2> and <3>.

Assume that $mD+2A_1$ is a part of the fixed components of $|A|$. Since $A = D+2A_1+F$, we have $h^0(\mathcal{O}_Z(F)) = h^0(\mathcal{O}_Z(A)) \geqq 2$ by Lemma 3.27. However since for a numerically effective line bundle L, $A\cdot L = 2$, $D\cdot L = 0$ and $A_1\cdot L>0$, we have $F\cdot L = 0$. It implies that every component of a divisor linearly equivalent to F is an exceptional curve of $\rho: Z \longrightarrow X$. Thus $h^0(\mathcal{O}_Z(F)) = 1$, which is a contradiction. Therefore this case <4> is reduced to other cases.

Here in all cases we have got a contradiction. Thus (1) implies (2).

It remains to show that $\bar{\Phi}$ is an embedding.

Let Y be the image of $\bar{\Phi}$. By assumption Y is a quartic surface. Assume that Y has the one-dimensional singular locus S.

Let H be a general hyperplane in $\mathbb{P}^3$. The intersection $Y \cap H$ has singularities at $S \cap H$. The arithmetic genus of $Y \cap H$ is $(4-1)(4-2)/2 = 3$. Now let $C \subset Z$ be the strict inverse image of $Y \cap H$. $\phi|_C : C \longrightarrow Y \cap H$ is a birational morphism. We have $p_a(C) \leq p_a(Y \cap H) = 3$ and the equality holds if and only if $\phi|_C$ is an isomorphism. On the other hand since any general member of $|L|$ is irreducible by Proposition 3.3, we have $C \in |L|$. Moreover C is smooth by the Bertini theorem. Thus $\phi|_C$ is not an isomorphism and we have $p_a(C)<3$. However by the adjunction formula $p_a(C) = (L^2 - D\cdot L)/2+1 = 3$, which is a contradiction. One sees that the singular locus of Y is 0-dimensional.

Note that every local ring of Y is Cohen-Macaulay of dimension ≤ 2 since Y is a hypersurface. The singular locus of Y has codimension ≥ 2. Thus by the Serre's criterion of normality (Cf. Matsumura [12]) the local ring $\mathcal{O}_{Y,y}$ is normal for every $y \in Y$. The morphism $X \longrightarrow Y$ is a birational finite one to a normal variety and therefore it is an isomorphism. Q.E.D.

Theorem 3.28. Let L be a polarization of degree 2 on a rational surface Z with an irreducible effective anti-canonical divisor D. The following conditions are equivalent.

(1) The rational map ϕ_L associated to L defines a surjective morphism of degree 2 to $\mathbb{P}^2$.

(2) The linear system $|L|$ has no fixed components.

(3) There exists no element $M \in \mathrm{Pic}(Z)$ with $M^2 = 0$, $M\cdot L = 1$ and

$M|_D \cong \mathcal{O}_D$.

Besides if one of the above equivalent conditions holds, then with the induced morphism $\bar{\Phi}: X \longrightarrow \mathbb{P}^2$ by ϕ_L, X has the structure of the branched double covering of $\mathbb{P}^2$ branching along a reduced sextic curve B.

Proof. First we show (3)$\Rightarrow$(2). Assume that $|L|$ has fixed components. Then $|L|$ contains a divisor $kF+\Gamma$ where k is an positive integer, F is an irreducible smooth elliptic curve with $F^2 = 0$, $F \cdot D = 0$, Γ is an irreducible smooth rational curve with $\Gamma^2 = -2$, $\Gamma \cdot D = 0$, $\Gamma \cdot F = 0$, by Proposition 3.7. Since $(kF+\Gamma)^2 = 2$, we have $k = 2$. Set $M = \mathcal{O}_Z(F+\Gamma)$. This M satisfies the conditions in (3). Thus (3) does not hold.

The implication (2)$\Rightarrow$(1) follows from Proposition 3.18.

Next we show (1)$\Rightarrow$(3). Assume that there exists $M \in \mathrm{Pic}(Z)$ with $M^2 = 0$, $M \cdot L = 1$ and $M|_D \cong \mathcal{O}_D$. We will deduce a contradiction under the assumption that ϕ_L is a morphism. By the same reason as in the proof of Theorem 3.25 one sees that the linear system $|M|$ is not empty. Let $A \in |M|$ and set

$$A = mD + \sum_{i=1}^{k} n_i A_i + F$$

where k, m, n_1, $\quad$, n_k are integers with $k \geq 0$, $m \geq 0$ and $n_j \geq 1$ $(1 \leq j \leq k)$, F is an effective divisor with $\mathrm{Supp}\, F \cap D = \phi$, and A_1, $\quad$, A_k are mutually different irreducible curves with $A_i \neq D$ and $A_i \cdot D > 0$ for $1 \leq i \leq k$. Now we have $A_i \cdot L > 0$ for every i by the same reason as in Theorem 3.25. Since

$$1 = M\cdot L = mD\cdot L + \sum_{i=1}^{k} n_i A_i\cdot L + F\cdot L$$

only one of the following two cases takes place.

<1> $k = 0$

<2> $k = 1$, $n_1 = 1$, $L\cdot A_1 = 1$ and $F\cdot L = 0$

Note that condition $k = 0$ is equivalent to that $m = 0$ because $0 = mD^2 + \sum_{i=1}^{k} n_i A_i\cdot D$, $A_i\cdot D \neq 0$ and $D^2 \neq 0$. The case <2> is reduced to <1> by Lemma 3.26. Thus we can assume that $A = F$, namely $\mathrm{Supp}\ A \cap D = \phi$. Let Δ be the fixed component of $|A|$ and $C = A-\Delta$. By Lemma 3.27 $C \neq 0$ and $C^2 \geqq 0$ since it is the moving part. For the moment we assume $C^2 = 0$. By Proposition 3.3 there is a smooth elliptic curve G with $G \cap D = \phi$ and an integer p with $C \in |pG|$. If $G\cdot L = 0$, then $G^2 < 0$ by the Hodge index theorem. By the adjunction foumula $p_a(G) = (G^2 - G\cdot D)/2+1 = (G^2/2)+1 \leqq 0$, which is a contradiction since G is an elliptic curve. Thus $G\cdot L > 0$. We have $p = 1$, $G\cdot L = 1$ and $\Delta\cdot L = 0$ since $1 = M\cdot L = pG\cdot L + \Delta\cdot L$. Thus $\phi|_G$: $G \longrightarrow \mathbb{P}^2$ is a generically one-to-one morphism and its image is a line in $\mathbb{P}^2$. We have $p_a(G) \leqq 0$, a contradiction again. Next we treat the case $C^2 > 0$. By Proposition 3.3, we can assume that C is an irreducible curve with $p_a(C) = (C^2/2)+1 \geqq 2$. By the same reason as just the above, one has $C\cdot L = 1$. Thus $\phi|_C$: $C \longrightarrow \mathbb{P}^2$ is a generically one-to-one morphism to a line. We have $p_a(C) \leqq 0$, a contradiction.

Thus conditions (1), (2) and (3) are equivalent.

Now we show the latter half of the theorem. By the Kawamata-Ramanujam vanishing theorem one sees easily that $h^1(mL) = 1$ and

$h^2(mL) = 0$ for any positive integer m. By Riemann-Roch we have $h^0(mL) = m^2+2$. Let u_1, u_2, u_3 be a basis of $H^0(L)$. Let S_m be the subspace of $H^0(mL)$ generated by monomials of u_i's of degree m. Since ϕ_L is a surjective morphism to $\mathbb{P}^2$, there is no non-zero homogeneous polynomial $P(U_1, U_2, U_3)$ with $P(u_1, u_2, u_3) = 0$. Thus $\dim_{\mathbb{C}} S_m = (m+2)(m+1)/2$. One sees that $H^0(L) = S_1$, $H^0(2L) = S_2$ and that there is a non-zero element $w \in H^0(3L)$ such that $H^0(3L)$ is a direct sum of $\mathbb{C}w$ and S_3. Let $\Phi\colon Z \longrightarrow \mathbb{P}(1, 1, 1, 3)$ be the morphism to the weighted projective space defined by $z \longmapsto (u_1(z), u_2(z), u_3(z), w(z))$. Let Y be its image. Note that since u_i's do not vanish simultaneously on Z, the image Y does not contain the point $(0, 0, 0, 1)$. Thus the composition $\pi\Phi$ with the projection $\mathbb{P}(1, 1, 1, 3)-\{(0, 0, 0, 1)\} \longrightarrow \mathbb{P}(1, 1, 1) = \mathbb{P}^2$ has the meaning and $\pi\Phi = \phi_L$ by definition. Moreover we can show that $\Phi\colon Z \longrightarrow \mathbb{P}(1, 1, 1, 3)$ factors through $\rho\colon Z \longrightarrow X$ by the same reason as in Proposition 3.20. Let $\bar{\Phi}\colon X \longrightarrow Y \subset \mathbb{P}(1, 1, 1, 3)$ be the induced morphism.

Lemma 3.29 If $P(u_1, u_2, u_3)+wQ(u_1, u_2, u_3) = 0$ for homogeneous polynomials $P(U_1, U_2, U_3)$, $Q(U_1, U_2, U_3)$ with $\deg P = \deg Q+3$, then $P = Q = 0$.

Proof. First assume that P and Q has a common non-constant divisor R. Set $P_1 = P/R$ and $Q_1 = Q/R$. They are homogeneous polynomials with $\deg P_1 = \deg Q_1+3$. Moreover under the assumption of the lemma we have $P_1(u_1, u_2, u_3)+wQ_1(u_1, u_2, u_3) = 0$ since $R(u_1, u_2, u_3) \neq 0$. Thus one sees that one can assume that P and Q has no non-constant common divisor and that one of P and Q is non-zero. Then the polynomial $P(U_1, U_2, U_3)+WQ(U_1, U_2, U_3)$ is irreducible and non-zero. Besides its zero-locus $Y' = \{ (a_1, a_2, a_3, b) \in P(1, 1, 1, 3) \mid P(a_1, a_2, a_3) + bQ(a_1, a_2, a_3) = 0 \}$ is irreducible. We have $Y = Y'$ since $Y \subset Y'$ by definition. However we have $(0, 0, 0, 1) \in Y = Y'$, which is a contradiction.

Q.E.D.

By the above lemma and by dimensional reasons one sees that $H^0(4L) = S_4+wS_1$, $H^0(5L) = S_5+wS_2$ and $H^0(6L) = S_6+wS_3$. (Here + denotes a direct sum.) Now since $w^2 \in H^0(6L)$, there are homogeneous polynomial P of degree 6 and Q of degree 3 such that

$$w^2+wQ(u_1, u_2, u_3)+P(u_1, u_2, u_3) = 0.$$

By replacing w by $w-Q(u_1, u_2, u_3)/2$, we can assume moreover that $Q = 0$. Here by construction Y agrees with the hypersurface in $P(1, 1, 1, 3)$ defined by $W^2-P(U_1, U_2, U_3) = 0$, which is nothing but the branched double covering branching along the sextic curve B ; $P(U_1, U_2, U_3) = 0$.

It remains to show that $\bar{\Phi}: X \longrightarrow Y$ is an isomorphism. Note that every local ring of Y is Cohen-Macaulay since Y is a hypersurface of a smooth manifold $\mathbb{P}(1, 1, 1, 3)-\{(0, 0, 0, 1)\}$. Thus it suffices to show that the singular locus S of Y is 0-dimansional by the same reason as in the proof of Theorem 3.25. It is equivalent to that B is reduced by Lemma 1.5. Now let H be a general line in $\mathbb{P}^2$. The inverse image $\pi^{-1}(H)$ by $\pi\colon Y \longrightarrow \mathbb{P}^2$ has singularities at $\pi^{-1}(H) \cap S$. The arithmetic genus of $\pi^{-1}(H)$ is $(\pi^*(H)^2+\omega_Y\cdot\pi^*(H))/2 + 1 = 2$. Let $C \subset Z$ be the strict inverse image of $\pi^{-1}(H)$ by Φ. $\Phi|_C\colon C \longrightarrow \pi^{-1}(H)$ is a birational morphism. We have $p_a(C) \leqq p_a(\pi^{-1}(H)) = 2$ and the equality holds if and only if $\Phi|_C$ is an isomorphism. However $C \in |L|$ and C is smooth. Thus $p_a(C) \leqq 1$ if $\dim S \geqq 1$. On the other hand we have $p_a(C) = (L^2-L\cdot D)/2+1 = 2$ and thus $\dim S = 0$. Q.E.D.

Before concluding this section we would like to give one more proposition and a lemma. The next lemma is due to Looijenga. We omit the proof here. (Cf. Looijenga [10])

Lemma 3.30. (Looijenga) Let A be an irreducible curve on Z with $A \cap D = \phi$ and $A^2 = -2$. Then $\mathcal{O}_Z(A) \in \mathrm{Pic}(Z)$ is an effective nodal root.

Remark. Since the conditions $\alpha^2 = -2$ and $\alpha\cdot\omega_Z = 0$ for $\alpha \in \mathrm{Pic}(Z)$ do not imply that α is a root, this lemma is not a trivial one.

Proposition 3.31. Let $\tilde{S} \subset \mathrm{Pic}(Z)$ be the set of nodal roots orthogonal to the polarization L. Then $\tilde{S}$ is a root system. Moreover singularities on X are a unique point with positive geometric genus at $w = \rho(D) \in X$ plus configuration of rational double points consisted of p_k of A_k-points, q_ℓ of D_ℓ-points, and r_m of E_m-points ($k \geq 1$, $\ell \geq 4$, $m = 6, 7, 8$) if and only if $\tilde{S}$ is isomorphic to the direct sum of p_k of irreducible root systems of type A_k for every k, q_ℓ of ones of type D_ℓ for every ℓ and r_m of ones of type E_m for $m = 6, 7, 8$. Here $\rho\colon Z \longrightarrow X$ is the contraction defined just after Lemma 3.19.

Proof. Let R be the set of all roots in $\mathrm{Pic}(Z)$. It is obvious by definition that $(\tilde{S}+\tilde{S}) \cap R \subset \tilde{S}$ and $\tilde{S} = -\tilde{S}$. And the orthgonal complement of L in $\mathrm{Pic}(Z)$ is negative-definite. Thus the former half of the proposition follows from the definition of the root system. (Cf. Bourbaki [3])

Let us proceed to the latter half. Let $\underline{\underline{E}}$ be the union of exceptional curves of $\rho\colon Z \longrightarrow X$. Let $\underline{\underline{E}}'$ be the union of D and the support of effective nodal roots orthogonal to L. In view of Lemma 2.1, it suffices to show that $\underline{\underline{E}} = \underline{\underline{E}}'$.

Let A be an irreducible curve on Z such that $\rho(A)$ is a point. If $A = D$, then $A \subset \underline{\underline{E}}'$ by definition. Assume $A \neq D$. By Lemma 3.19, we have $A^2 = -2$ and $A \cap D = \phi$. By Lemma 3.30, we have $A \subset \underline{\underline{E}}'$. Thus $\underline{\underline{E}} \subset \underline{\underline{E}}'$. Conversely let A be an irreducible component of $\underline{\underline{E}}'$. If $A = D$, then $A \subset \underline{\underline{E}}$ by Lemma 3.19. Assume $A \neq D$. There exists an effective divisor $\sum n_i A_i$ ($0 < n_i \in Z$, A_i is an irreducible curve.) containing A as a component such that $\mathcal{O}_Z(\sum n_i A_i) \in \mathrm{Pic}(Z)$ is a nodal root orthogonal to L. We may assume

$A = A_1$. It follows that $A_i \cdot L = 0$ for every i from $\Sigma n_i A_i \cdot L = 0$ since L is numerically effective. By Lemma 3.19 we have $A = A_1 \subset \underline{E}$. Thus $\underline{E} = \underline{E}'$. Q.E.D.

Now according to Theorem 3.25 and Theorem 3.28 we can decide whether Z represents a reduced sextic curve or a normal quartic surface by studying the morphism $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(D)$. Proposition 3.31 shows that the morphism $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(D)$ contains information about singularities on the objects we are considering. Therefore if we had a criterion written with group-theoretic words about $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(D)$ by which we can decide $L \in \mathrm{Pic}(Z)$ is a polarization or not, then classification of all singularities of objects under consideration would be accomplished.

In the next section, we show that this is the case when $t = 9-\omega_Z^2 = 10$.

§ 4. Determination of the polarization class (when $t = 10$)

In section 1, 2 and 3, we only assumed that $t = 9-\omega_Z{}^2 \geq 3$. In section 4 restriction appeared; existence of polarization implies $t \geq 10$. However in this section and following ones, we restrict ourselves to the case $t = 10$. There are two reasons to do so. First if $t = 10$, we can easily determine all elements $\lambda \in P$ with $\lambda \cdot \kappa = 0$ and $\lambda^2 = 2$ or 4 compared with the case $t \geq 11$. Secondly we have a group-theoretic criteria by which we can decide $L \in \mathrm{Pic}(Z)$ with $L \cdot \omega_Z = 0$ and $L^2 = 2$ or 4 is a polarization or not.

In this section we always assume that $t = 10$ (i.e. $\omega_Z{}^2 = -1$) even if there is no mentioning.

Proposition 4.1. Assume that $\omega_Z{}^2 = -1$. (i.e. $t = 10$) An element $L \in \mathrm{Pic}(Z)$ with $L|_D \cong \mathcal{O}_D$ and $L^2 > 0$ is a polarization if and only if $L \in V_S \cap C_+$ where C_+ is a connected component of the positive cone $C = \{ x \in \mathrm{Pic}(Z) \otimes \mathbb{R} \mid x^2 > 0 \}$ containing ample line bundles and

$$V_S = \{ x \in \mathrm{Pic}(Z) \otimes \mathbb{R} \mid x \cdot \omega_Z = 0,\ x \cdot r \geq 0 \ \text{ for any effective nodal root } r \in \mathrm{Pic}(Z) \}.$$

Proof. 'Only if' part is trivial since L is numerically effective. To show 'if' part, we have to check conditions in Definition 3.1. The conditions (1) and (3) are obvious by assumption. We show (2), i.e., L is numerically effective. It suffices to show that for every irreducible curve A, the inequality $L \cdot A \geq 0$ holds.

Recall that the positive cone C has just two connected components. One is C_+. The other is $C_- = -C_+$.

If $A^2>0$, the restriction to the orthogonal complement $(\mathbb{R}A)^\perp$ of A in $\mathrm{Pic}(Z)\otimes\mathbb{R}$ of the intersection form is negative definite since the intersection form on $\mathrm{Pic}(Z)$ has signature $(1, 10)$. Thus $(\mathbb{R}A)^\perp \cap \bar{C} = \{0\}$. ($^-$ denotes the closure.) It implies that C_+ lies in a half space bounded by the hyperplane $(\mathbb{R}A)^\perp$. Since both L and any ample line bundle belongs to C_+, we have $L\cdot A>0$. Moreover by a similar argument we have $L\ A>0$ for any curve A with $A^2 = 0$. Here note that we did not use that A is irreducible until now. Assume that $A^2<0$. By the adjunction formula, one sees that there are three cases.

(i) $A = D$.

(ii) $A^2 = -2$ and $A \cap D = \phi$.

(iii) $A^2 = -1$ and $A\cdot D = 1$.

If $A = D$, then $L\cdot D = 0$ by assumption $L|_D \cong \mathcal{O}_D$. In case (ii), $\mathcal{O}_Z(A)$ is an effective nodal root by Lemma 3.30. Thus it follows from the assumption $L\in V_S$ that $A\cdot L = \mathcal{O}_Z(A)\cdot L\geq 0$. In order to manipulate case (iii), we need the assumption $D^2 = -1$. Set $C = A+D$. We have $C^2 = -1+2-1 = 0$. Thus by the above argument we have $L\ (A+D) = L\cdot A>0$. We obtain not only numerical effectiveness but also condition (4) in Definition 3.1. Q.E.D.

Next we determine elements $\lambda\in P = \mathbb{Z}\varepsilon_0+\mathbb{Z}\varepsilon_1+ \quad +\mathbb{Z}\varepsilon_{10}$ with $\lambda^2 = 2$ or 4 and $\lambda\ \kappa = 0$ up to the action of the Weyl group W.

Here $\kappa = -3\varepsilon_0+\varepsilon_1+ \cdots +\varepsilon_{10}$. Let Γ be the orthogonal complement of $\mathbb{Z}\kappa$ in P. We denote

$$\mho = \{ x\in\Gamma\otimes\mathbb{R} \mid x^2>0 \}$$

$$\mho_+ = \{ x\in\mho \mid x\cdot\varepsilon_0>0 \}$$

$$\mho_- = \{ x\in\mho \mid x\cdot\varepsilon_0<0 \}.$$

It is easy to see that $\mho_\pm$ are conected components of $\mho$ and $\mho = \mho_+ \cup \mho_-$. Moreover we denote

$$\mathfrak{V} = \{ x\in\Gamma\otimes\mathbb{R} \mid x\cdot\gamma_i\geqq 0 \quad \text{for} \quad 1\leqq i\leqq 10 \}$$

where $\gamma_1 = \varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3$, $\gamma_i = \varepsilon_{i-1}-\varepsilon_i$ for $2\leqq i\leqq 10$. The following lemma is due to Looijenga. (Looijenga [10])

Lemma 4.2. $\mho_+ \subset W\mathfrak{V}$.

The rest of this section is devoted to verify the following.

Proposition 4.3. Assume $t = 10$. Any element $\lambda\in P$ with $\lambda^2 = 4$ and $\lambda\cdot\kappa = 0$ is conjugate to one of the following elements with respect to the action of W.

$$\pm(9\varepsilon_0-3\varepsilon_1-3\varepsilon_2-3\varepsilon_3-3\varepsilon_4-3\varepsilon_5-3\varepsilon_6-3\varepsilon_7-3\varepsilon_8-2\varepsilon_9-\varepsilon_{10})$$

$$\pm(7\varepsilon_0-3\varepsilon_1-2\varepsilon_2-2\varepsilon_3-2\varepsilon_4-2\varepsilon_5-2\varepsilon_6-2\varepsilon_7-2\varepsilon_8-2\varepsilon_9-2\varepsilon_{10})$$

Proposition 4.4. Assume $t = 10$. Any element $\lambda\in P$ with $\lambda^2 = 2$ and $\lambda\cdot\kappa = 0$ is conjugate to one of the following elements with respect to the action of W.

$$\pm(6\varepsilon_0-2\varepsilon_1-2\varepsilon_2-2\varepsilon_3-2\varepsilon_4-2\varepsilon_5-2\varepsilon_6-2\varepsilon_7-2\varepsilon_8-\varepsilon_9-\varepsilon_{10})$$

Proof of Proposition 4.3.

If λ belongs to $\tilde{U}_-$, then obviously $-\lambda$ belongs to $\tilde{U}_+$, $(-\lambda)^2 = 4$ and $(-\lambda)\cdot\kappa = 0$. Besides every element in $\tilde{U}_+$ is conjugate to an element in $\tilde{V}$ by Lemma 4.3. Thus we have only to show that the following system of equalities and inequalities hold for integers $x, y_1, \cdots, y_{10}$ if and only if $(x, y_1, \cdots, y_{10}) = (9, 3, \cdots, 3, 2, 1)$ or $(7, 3, 2, \cdots, 2)$.

$$(4.1) \qquad \begin{cases} x^2 = \sum_{i=1}^{10} y_i^2 + 4 \\ 3x = \sum_{i=1}^{10} y_i \\ x \geqq y_1+y_2+y_3 \\ y_1 \geqq y_2 \geqq y_3 \geqq y_4 \geqq y_5 \geqq y_6 \geqq y_7 \geqq y_8 \geqq y_9 \geqq y_{10} \end{cases}$$

We need several steps.

STEP 1.

Lemma 4.5. If (4.1) holds, then $x \geqq 7$ and $y_i > 0$ for $1 \leqq i \leqq 10$.

Proof. By the Schwartz inequality we have for $1 \leqq \alpha \leqq 10$ $(3x-y_\alpha)^2 = (\sum_{i\neq\alpha} y_i)^2 \leqq 9(x^2-y_\alpha^2-4)$. Thus $5(y_\alpha - \frac{3}{10}x)^2 - \frac{9}{20}x^2 + 18 \leqq 0$. One sees that $x \neq 0$ and that $y_\alpha > 0$ or < 0 according as $x > 0$ or < 0. Assume $x < 0$. We have $y_{10} < 0$. It implies that $3x \geqq 3(y_1+y_2+y_3) \geqq \sum_{j=1}^{9} y_j > \sum_{j=1}^{10} y_j = 3x$, a contradiction. Therefore $x > 0$ and $y_\alpha > 0$ for $1 \leqq \alpha \leqq 10$. Moreover by the Schwartz inequality we have $9x^2 = (\sum y_i)^2 \leqq 10(\sum y_i^2) = 10(x^2-4)$. Thus $x \geqq 7$. Q.E.D.

Lemma 4.6. If (4.1) holds and if $x \leqq 10$, then $(x, y_1, \quad , y_{10}) = (9, 3, \quad , 3, 2, 1)$ or $(7, 3, 2, \quad , 2)$.

Proof. We can assume $7 \leqq x \leqq 10$ by Lemma 4.5. First assume $x = 7$. By the Schwartz inequality we have $(21-y_1)^2 \leqq 9(45-y_1^2)$. It implies $5y_1^2-21y_1+18 \leqq 0$ and thus $0 < y_1 \leqq 3$. If $y_1 = 3$, then $y_2+ \quad + y_{10} = 18$ and $y_2^2+ \quad +y_{10}^2 = 36$. Since $18^2 = 9 \times 36$, the equality in the Schwartz inequality $(\sum_{i \geqq 2} y_i)^2 \leqq 9(\sum_{i \geqq 2} y_i^2)$ holds. Thus $y_2 = \quad = y_{10} = 2$. We have the solution $(7, 3, 2, \quad , 2)$. If $y_1 \leqq 2$, then $21 = y_1+y_2+ \quad +y_{10} \leqq 20$, which is a contradiction. Secondly assume $x = 8$. We can show similarly that there is no solution in this case. Thirdly assume $x = 9$. By the Schwartz inequality we have $5y_1^2-27y_1+18 \leqq 0$. Thus $0 < y_1 \leqq 4$. Assume $y_1 = 4$. We have $(23-y_2)^2 = (y_3+ \quad +y_{10})^2 \leqq 8 \times \sum_{i \geqq 3} y_i^2 = 8(61-y_2^2)$, which implies $y_2 \leqq 3$. If $y_2 \leqq 2$, then $23 = \sum_{i \geqq 2} y_i \leqq 18$, a contradiction. Thus $y_2 = 3$. Since $y_1+y_2+y_3 \leqq x = 9$ we have moreover $y_3 \leqq 2$. We have $20 = \sum_{i \geqq 3} y_i \leqq 16$, a contradiction again. Thus $0 < y_1 \leqq 3$. Now we assume that k of $\{y_1, y_2, \quad , y_{10}\}$ are 3, ℓ of them are 2 and m of them are 1. We have $k+\ell+m = 10$, $3k+2\ell+m = 27$ and $9k+4\ell+m = 77$. One sees easily that $k = 8$, $\ell = 1$ and $m = 1$. We have the solution $(9, 3, 3, \quad , 2, 1)$. Lastly assume $x = 10$. Similarly we see that there is no solution in this case. Q.E.D.

STEP 2.

Next we set

$$x = 3z+\varepsilon,\ y_i = z+\delta_i\ (1\leqq i\leqq 9),\ y_{10} = \delta_{10}.$$

Equalities and inequalities (4.1) are equivalent to the next ones.

$$(4.2)\quad \begin{cases} \langle 1\rangle & \varepsilon\geqq\delta_1+\delta_2+\delta_3 \\ \langle 2\rangle & \delta_1\geqq\delta_2\geqq \qquad \geqq\delta_9 \\ \langle 3\rangle & z+\delta_9\geqq\delta_{10} \\ \langle 4\rangle & \delta_{10}>0 \\ \langle 5\rangle & \delta_1+\delta_2+ \qquad +\delta_{10} = 3\varepsilon \\ \langle 6\rangle & 2z(\delta_1+\delta_2+ \qquad +\delta_9)+(\delta_1{}^2+\delta_2{}^2+ \qquad \delta_9{}^2)+\delta_{10}{}^2 \\ & \qquad\qquad = 6\varepsilon z+\varepsilon^2-4 \end{cases}$$

Lemma 4.7. If ε, δ_1, $\qquad$, δ_{10} are 0 or ±1, then the solution of (4.2) is $z = 3$, $\varepsilon = 0$, $\delta_1 = \delta_2 = \qquad = \delta_8 = 0$, $\delta_9 = -1$, $\delta_{10} = +1$.

Proof. By $\langle 4\rangle$ we have $\delta_{10} = 1$. First assume $\varepsilon = 0$. If $\delta_1 = 1$, then by $\langle 1\rangle$. $\langle 2\rangle$ we have only two cases; (a) $\delta_2 = 0$, $\delta_3 = \qquad = \delta_9 = -1$, (b) $\delta_2 = \delta_3 = \qquad = \delta_9 = -1$. In both cases $\langle 5\rangle$ does not hold. If $\delta_1 = 0$, then by $\langle 2\rangle$. $\langle 5\rangle$ $\delta_2 = \qquad = \delta_8 = 0$, $\delta_9 = -1$. Substituting them to $\langle 6\rangle$, we have $z = 3$. Thus $\langle 3\rangle$ is also satisfied. We have the desired solution. If $\delta_1 = -1$, by $\langle 2\rangle$ $\delta_2 = \qquad = \delta_9 = -1$. They do not satisfy $\langle 5\rangle$. Secondly assume $\varepsilon = +1$. If $\delta_1\leqq 0$, then by $\langle 2\rangle$, $\langle 5\rangle$ $1 \geqq \delta_1+\delta_2+ \qquad +\delta_{10} = 3$, which is a contradiction. Thus $\delta_1 = 1$. By $\langle 1\rangle$, $\langle 2\rangle$ we have only three cases.

(c) $\delta_2 = 1$, $\delta_3 = \cdot = \delta_9 = -1$, (d) $\delta_2 = \delta_3 = 0$, $\delta_4, \delta_5, \cdot, \delta_9 \leqq 0$ (e) $\delta_2 = 0$, $\delta_3 = \cdot = \delta_9 = -1$. In any case <5> does not hold. Thirdly assume $\varepsilon = -1$. If $\delta_1 = 1$, then $\delta_2 = \delta_3 = -1$ by <1>. By <2> we have moreover $\delta_4 = \quad = \delta_9 = -1$. In this case <5> does not hold. If $\delta_1 = 0$, then there are only two cases by <1>, <2>. (f) $\delta_2 = 0$, $\delta_3 = \quad = \delta_9 = -1$ (g) $\delta_2 = \delta_3 = \quad = \delta_9 = -1$. Anyway <5> does not hold. If $\delta_1 = -1$, then we have $\delta_2 = \quad = -1$ by <2> and <5> does not hold. Q.E.D.

Lemma 4.8. Assume one of $\varepsilon, \delta_1, \quad, \delta_{10}$ is ± 2, at most one of them is ± 1 and the rest are 0. Then (4.2) has no solution.

Proof. First assume $\varepsilon = \pm 2$. By <4> we have $\delta_{10} = 1$. By assumption we have $\delta_1 = \cdot = \delta_9 = 0$. Then <5> does not hold. Secondly assume $\varepsilon = \pm 1$. By <4> we have $\delta_{10} = 2$. By assumption one sees $\delta_1 = \quad = \delta_9 = 0$. Then <5> doen not hold. Thirdly assume $\varepsilon = 0$. We have 3 cases: (a) $\delta_1 = \quad = \delta_8 = 0$, $\delta_9 = -2$, $\delta_{10} = 1$ (b) $\delta_1 = \quad = \delta_8 = 0$, $\delta_9 = -1$, $\delta_{10} = 2$ (c) $\delta_1 = \quad = \delta_8 = \delta_9 = 0$, $\delta_{10} = 2$. In any case <5> is not satisfied. Q.E.D.

By the next lemma we can complete the proof of Proposition 4.3.

Lemma 4.9. If an integral solution of (4.1) satisfies $x \geqq 11$, then there exist integers $z, \varepsilon, \delta_1, \quad, \delta_{10}$ satisfying $x = 3z+\varepsilon$, $y_i = z+\delta_i$ $(1 \leqq i \leqq 9)$, $y_{10} = \delta_{10}$, equalities and inequalities

(4.2) and $\varepsilon^2 + \sum_{i=1}^{10} \delta_i^{\,2} \leqq 5$.

Since inequality $\varepsilon^2 + \sum \delta_i^{\,2} \leqq 5$ implies that one of the assumptions in Lemma 4.7 and 4.8 is satisfied, it follows from Lemma 4.7, 4.8 and 4.9 that (4.1) has no solution with $x \geqq 11$. Thus by Lemma 4.6 we have Proposition 4.3. Q.E.D.

STEP 3.

Now we have to show Lemma 4.9. Here we introduce an Euclidean metric $(\ ,\ )$ on $P\otimes\mathbb{R}$ by $(\varepsilon_i, \varepsilon_i) = 1$ $(0 \leqq i \leqq 10)$ and $(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$. By this metric we can define the distance dist(A, B) of two subsets $A, B \subset P\otimes\mathbb{R}$. Let P_i denote the orthogonal complement of the set $\{\varkappa, \gamma_1, \gamma_2, \quad , \gamma_{10}\} - \{\gamma_i\}$ in $P\otimes\mathbb{R}$ with respect to the intersection form, i.e., $P_i = \{ x \in P\otimes\mathbb{R} \mid x\cdot\varkappa = 0,\ x\cdot\gamma_j = 0 \text{ for } 1 \leqq j \leqq 10,\ j \neq i \}$. Set $T_c = \{ x \in P\otimes\mathbb{R} \mid x\cdot\varkappa = 0,\ x\cdot x = c,\ x\cdot\gamma_i \geqq 0 \text{ for } 1 \leqq i \leqq 10 \} \subset \Gamma\otimes\mathbb{R}$, $H_g = \{ x \in P\otimes\mathbb{R} \mid x\cdot\varepsilon_0 \geqq g \}$ where c, g are positive real numbers. We would like to show that $T_4 \cap H_{11}$ lies too near to P_{10} to have lattice points on it. We need further several lemmas.

The following one treats a general situation.

Lemma 4.10. Let F be a three dimensional real vector space equipped with an intersection form $\langle\ ,\ \rangle$ of signature (1, 2) and with a positively definite inner product $(\ ,\ )$. Let L be a line in F passing through the origin. For a positive real number a

we set $Q = \{ x \in F \mid \langle x, x\rangle = a \}$. Let $E \subset F$ be a two-dimensional linear subspace of F with $E \cap Q \neq \phi$. Then $E \cap Q$ has two connected component each of which is diffeomorphic to $\mathbb{R}$. Let ϕ: $\mathbb{R} \longrightarrow E \cap Q$ be a diffeomorphism to one connected component. Then for any closed interval $[b, c] \subset \mathbb{R}$ and for every $\lambda \in [b, c]$,

$$\mathrm{dist}(\phi(\lambda), L) \leqq \max \{ \mathrm{dist}(\phi(b), L), \mathrm{dist}(\phi(c), L) \}.$$

Proof. Since the restriction of the intersection form $\langle\ ,\ \rangle$ to E has signature $(1, 1)$, $E \cap Q$ is a hyperbolic curve. Therefore $E \cap Q$ is diffeomorphic to two copies of $\mathbb{R}$. We divide the rest of the proof into two cases.

<u>Case 1.</u> $L \subset E$.

Figure 4.1.

For every non-negative real number $e \in \mathbb{R}$, set $D_e = \{ x \in E \mid \mathrm{dist}(x, L) \leqq e \}$. D_e is a closed connected set bounded by two lines parallel to L. Note that $D_e \cap \phi([b, c])$ is always connected. Set $d_0 = \mathrm{dist}(\phi(\lambda), L)$ and assume $d_0 > \max\{ \mathrm{dist}(\phi(b), L), \mathrm{dist}(\phi(c), L) \}$. There exists a sufficiently small positive real number $\varepsilon > 0$ such that $D_{d_0 - \varepsilon} \ni \phi(b), \phi(c)$. Since $D_{d_0-\varepsilon} \cap \phi([b, c])$ is connected, $D_{d_0-\varepsilon} \cap \phi([b, c]) = \phi([b, c])$. It implies $\phi(\lambda) \in D_{d_0-\varepsilon}$. We have $d_0 = \mathrm{dist}(\phi(\lambda), L) \leqq$

$d_0 - \varepsilon$, a contradiction.

<u>Case 2.</u> $L \not\leq E$.

Similarly we set for non-negative real number $e \in \mathbb{R}$, $D_e = \{ x \in E \mid \mathrm{dist}(x, L) \leqq e \}$. In this case D_e is the interior and the boundary of an oval. Since $D_e \cap \phi([b, c])$ is always connected, we get the desired inequality by the same reason as in Case 1. Q.E.D.

Figure 4.2.

We now return to our case. For every subset $I \subset \{ 1, 2, 3, , 10 \}$, we set $P_I = (\bigcap_{i \in I^c} F_i) \cap (\mathbb{R}\kappa)^{\perp}$ where I^c is the complement of I, F_i is the orthogonal complement of γ_i in $P \otimes \mathbb{R}$, and $(\mathbb{R}\kappa)^{\perp}$ is the orthogonal complement of κ. Note that $P_{\{i\}} = P_i$. Next we define linear functions u, v_1, , v_{10}: $P \otimes \mathbb{R} \longrightarrow \mathbb{R}$ by $u(x) = x \cdot \varepsilon_0$ and $v_i(x) = x \cdot \gamma_i$ for $1 \leqq i \leqq 10$. By direct calculation we obtain;

Lemma 4.11. $P_i \cap T_4$ is a unique point for $1 \leqq i \leqq 9$ and we have $u(x_i) < 11$ for $\{x_i\} = P_i \cap T_4$, $1 \leqq i \leqq 9$. $P_{10} \cap T_4$ is empty. (Indeed $\max\{ u(x_i) \mid 1 \leqq i \leqq 9 \} = u(x_9) = 6 \times \sqrt{2}$.)

The next lemma is the key part of this section.

Lemma 4.12. For every subset $I \subset \{1, 2, \cdots, 10\}$ with $\# I \geqq 3$ and for every $x \in P_I \cap T_4 \cap H_{11}$, there exist a subset $J \subset I$ with $\# J = \# I - 1$ and a point $y \in P_J \cap T_4 \cap H_{11}$ with $\mathrm{dist}(y, P_{10}) \geqq \mathrm{dist}(x, P_{10})$.

Proof. First note that unless $I = \{10\}$ or $I = \phi$, the restriction of the intersection form of $P \otimes \mathbb{R}$ to the space spanned by τ_i, $i \in \{1, 2, \cdots, 10\} - I$ is negatively definite. Thus the intersection form has signature $(1, k-1)$ on P_I unless $I = \{10\}$ or $I = \phi$ where $k = \# I$. Assume $k \geqq 3$. One sees easily that $P_I \cap T_4 \cap H_{10} \neq \phi$. Assume that there exists $i \in I$ with $v_i(x) = 0$ for $x \in P_I \cap T_4 \cap H_{11}$. Then $x \in P_{I-\{i\}} \cap T_4 \cap H_{11}$ and setting $J = I - \{i\}$, $y = x$ we get the lemma. Thus in what follows we assume that $v_i(x) \neq 0$ for every $i \in I$. Since $x \in T_4$, we have $v_i(x) > 0$ for $i \in I$. We denote $Q = \{ z \in P \otimes \mathbb{R} \mid z \cdot z = 4 \}$. $P_I \cap Q$ is a quadratic hypersurface spanning P_I. $P_I \cap Q$ has two connected components. Let $(P_I \cap Q)_0$ be the connected component of $P_I \cap Q$ containing x. Set $c_0 = \min \{ u(y) \mid y \in (P \cap Q)_0 \}$. We have $c_0 > 0$ and $c_0 < 11$ by Lemma 4.11. If $-c_0 < g < c_0$, then $P_I \cap Q \cap \partial H_g = \phi$. If $g = \pm c_0$, then $P_I \cap Q \cap \partial H_g$ is one point. If $|g| > c_0$, then $P_I \cap Q \cap \partial H_g$ is a smooth $(k-2)$-dimensional manifold. In particular $P_I \cap Q \cap \partial H_{u(x)}$ is a smooth $(k-2)$-dimensional manifold. Let S' be the tangent space of $P_I \cap Q \cap \partial H_{u(x)}$ at x. If $0 \in S'$, then $0 \in S' \subset \partial H_{u(x)}$ and $0 = u(0) = u(x) \geqq 11$. It is a contradiction. Thus $0 \notin S'$. Let $\hat{V} = \{ z \in P_I \mid v_i(z) \geqq 0$ for $i \in I \}$. $\hat{V}$ is a convex cone in P_I and x

belongs to the interior of $\hat{V}$. Since $\dim S' \geq 1$, S' intersects some wall of $\hat{V}$. i.e., $S' \cap (\hat{V} \cap P_{I-\{i_0\}}) \neq \phi$ for some $i_0 \in I$. Note that there exists $y_0 \in S' \cap (\hat{V} \cap P_{I-\{i_0\}})$ with $y_0 \cdot y_0 > 0$. Otherwise $S' \cap (\hat{V} \cap P_{I-\{i_0\}}) \subset P_{10}$ and moreover the tangent space S' of $P_I \cap Q \cap \partial H_{u(x)}$ at x intersects P_{10}, which is impossible. Thus such y_0 always exists. Let M' be the linear span of x and y_0. If $0 \in M'$, then $x \in M' \subset P_{I-\{i_0\}}$ and we have $v_{i_0}(x) = 0$, a contradiction. Let M be the linear span of x, y_0 and 0. It follows $\dim M = 2$. Since $x \in M$ and $x \cdot x = 4$, the restriction of the intersection form to M has signature $(1, 1)$. We have the following figure.

Figure 4.3.

Next we would like to show $(M \cap Q)_0 \subset H_{u(x)}$, i.e., $u(y) \geq u(x)$ for every $y \in (M \cap Q)_0$, where $(M \cap Q)_0$ is the connected component of $M \cap Q$ containing x. If $M \subset \partial H_{u(x)}$, we have nothing to prove. Thus we assume $M \not\subset \partial H_{u(x)}$. $M \cap \partial H_{u(x)}$ is a line containing x and y_0, that is, $M \cap \partial H_{u(x)} = M'$. Recall that M' is the tangent line of $M \cap Q$ at x by definition. Since $M \cap Q$ is a hyperbolic curve, $(M \cap Q)_0$ lies on one side of M'. We have either $u(y) \geq u(x)$ for every $y \in (M \cap Q)_0$ or

$0<u(y)\leq u(x)$ for every $y\in(M\cap Q)_0$. Since obviously $u(y)$ is unbounded on $(M\cap Q)_0$, we have $(M\cap Q)_0\subset H_{u(x)}$. Now $M\cap P_{I-\{i\}}$ is a line in M passing through the origin for every $i\in I$ since $P_{I-\{i\}}=\mathrm{Ker}v_i\cap P_I\not\ni x$. One sees that $M\cap T_4$ coincides with the closure of the connected component of $M\cap Q-\bigcup_{i\in I}M\cap P_{I-\{i\}}$ containing x. Since $y_0\in P_{I-\{i_0\}}$ and $y_0\cdot y_0>0$, $M\cap P_{I-\{i_0\}}$ intersects with $(M\cap Q)_0$. It implies that $M\cap T_4$ is a connected closed proper subset of $(M\cap Q)_0$. Thus we have $Y=\partial(M\cap T_4)\cap(\bigcup_{i\in I}M\cap P_{I-\{i\}})\neq\phi$. Pick $y_1\in Y$. There exists $i_1\in I$ with $y_1\in\partial(M\cap T_4)\cap P_{I-\{i_1\}}$. Set $J=I-\{i_1\}$. Then $y_1\in P_J\cap T_4$ and $y_1\in(M\cap Q)_0\subset H_{u(x)}\subset H_{11}$. Moreover by Lemma 4.10, $\mathrm{dist}(y_1, P_{10})\geq\mathrm{dist}(x, P_{10})$.

Q.E.D.

Lemma 4.13. For every subset $I\subset\{1, 2, 3, \quad , 10\}$ with $\#I=2$ and $10\notin I$, we have $P_I\cap H_{11}\cap T_4=\phi$.

Proof. Set $I=\{i, j\}$. Since $i\neq 10$, $j\neq 10$, we have $P_i-\{0\}, P_j-\{0\}\subset\{y\in P_I \mid y\cdot y>0\}$. Thus if $T_4\cap P_I$ is not empty, it is a compact connected arc contained in a hyperbolic curve. However, for a point y in $P_i\cap T_4$ and $P_j\cap T_4$, $u(y)<11$ by Lemma 4.11. Thus for every $y\in T_4\cap P_I$, $u(y)<11$. It implies $T_4\cap P_I\cap$

Figure 4.4.

$H_{11} = \phi$. Q.E.D.

Lemma 4.14. For a subset $I = \{ k, 10 \}$ with $1 \leqq k \leqq 9$, the function $P_I \cap T_4 \cap H_{11} \ni x \longrightarrow \mathrm{dist}(x, P_{10})$ attains its maximal value on the set $P_I \cap T_4 \cap \partial H_{11}$.

Figure 4.5.

Proof. Since $P_{10} \subset \{y \in P_I \mid y \cdot y = 0\}$ and $P_k - \{0\} \subset \{ y \in P_I \mid y \cdot y > 0 \}$, $P_I \cap T_4$ is an arc as in the left figure. Since $u(y_2) < 11$ for $y_2 \in P_k \cap T_4$, y_2 and the origin lie on the same side with respect to ∂H_{11}. It implies that there are not two connected components of $T_4 \cap P_I \cap H_{11}$ but there is only one. In view of the fact that $P_{10} \cap T_4$ is the asymptotic line of $T_4 \cap P_I \cap H_{11}$, one sees that the distance to P_{10} attains the maximal value at $T_4 \cap P_I \cap \partial H_{11}$ by Lemma 4.10. Q.E.D.

Lemma 4.15. The set $T_4 \cap P_{\{k,\ 10\}} \cap \partial H_{11}$ consists of a unique point $\{y_k\}$ for $1 \leqq k \leqq 9$. Besides we have $\mathrm{dist}(y_k, P_{10}) < 1$ for $1 \leqq k \leqq 9$.

Proof. The former half is trivial. By direct calculation we have

$$\max_k \mathrm{dist}(y_k, P_{10}) = \mathrm{dist}(y_9, P_{10}) = \sqrt{70}/9 < 1. \qquad \text{Q.E.D.}$$

Corollary 4.16. For every point $x \in T_4 \cap H_{11}$, $\mathrm{dist}(x, P_{10}) < 1$.

Proof of Lemma 4.9.

First note that the set $\{ z(3\varepsilon_0 - \sum_{i=1}^{9} \varepsilon_i) \mid z \in \mathbb{Z} \}$ exhausts the lattice points (points whose coordinates are all integers) on P_{10}. The minimum distance of lattice points on P_{10} is $\sqrt{18}$. Thus for every point $x \in P_{10}$ there exists a lattice point $w \in P_{10}$ with $\mathrm{dist}(x, w) \leqq \sqrt{18}/2$.

Let $y_0 \in T_4 \cap H_{11}$ be an arbitrary lattice point. Let $x_0 \in P_{10}$ be the point on P_{10} which attains the distance between y_0 and P_{10}, i.e., $\mathrm{dist}(y_0, P_{10}) = \mathrm{dist}(y_0, x_0)$. The line passing through x_0 and y_0 is perpendicular to P_{10}. Let $w_0 \in P_{10}$ be the lattice point with $\mathrm{dist}(x_0, w_0) \leqq \sqrt{18}/2$. By the Pythagorean theorem and by Corollary 4.16 $\mathrm{dist}(y_0, w_0)^2 < 18/4+1 = 5.5$. Since $\mathrm{dist}(y_0, w_0)^2$ is an integer, we have $\mathrm{dist}(y_0, w_0)^2 \leqq 5$, which is the desired result. Q.E.D.

By the same method we can also verify Proposition 4.4. Indeed it is easy to check the following lemmas.

Lemma 4.10. The system of equalities and inequalities

$$\left\{ \begin{array}{l} x^2 = \sum_{i=1}^{10} y_i^2 + 2 \\ 3x = \sum_{i=1}^{10} y_i \\ x \geqq y_1 + y_2 + y_3 \end{array} \right. \tag{4.3}$$

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_{10}$$

is satisfied by integers $x, y_1, \cdots, y_{10}$ with $x \leq 10$ if and only if $(x, y_1, \cdots, y_{10}) = (6, 2, 2, \cdots, 2, 1, 1)$.

Lemma 4.11. (1) For every point $y \in T_2 \cap H_{11}$, $\mathrm{dist}(y, P_{10}) < 1$.

(2) If an integral solution of (5.3) satisfies $x \geq 11$, then there exist integers $z, \varepsilon, \delta_1, \cdots, \delta_{10}$ satisfying

$$(4.4) \quad \begin{cases} \langle 1\rangle & \varepsilon \geq \delta_1+\delta_2+\delta_3 \\ \langle 2\rangle & \delta_1 \geq \delta_2 \geq \cdots \geq \delta_9 \\ \langle 3\rangle & z+\delta_9 \geq \delta_{10} \\ \langle 4\rangle & \delta_{10} > 0 \\ \langle 5\rangle & \delta_1+\delta_2+\cdots+\delta_{10} = 3\varepsilon \\ \langle 6\rangle & 2z(\delta_1+\delta_2+\cdots+\delta_9)+(\delta_1{}^2+\delta_2{}^2+\cdots+\delta_9{}^2)+\delta_{10}{}^2 \\ & \qquad = 6\varepsilon z+\varepsilon^2-2 \\ \langle 7\rangle & \varepsilon^2+\sum_{i=1}^{10} \delta_i{}^2 \leq 5 \end{cases}$$

such that $x = 3z+\varepsilon$, $y_i = z+\delta_i$ $(1 \leq i \leq 9)$, $y_{10} = \delta_{10}$.

Lemma 4.12. If $\varepsilon, \delta_1, \cdots, \delta_{10}$ are 0 or ± 1, then the solution of (4.4) is $z = 2$, $\varepsilon = 0$, $\delta_1 = \cdots = \delta_8 = 0$, $\delta_9 = -1$, $\delta_{10} = 1$.

Lemma 4.13. Assume that one of $\varepsilon, \delta_1, \cdots, \delta_{10}$ is ± 2, at most one of them is ± 1, and the rest are 0. Then (4.4) has no solution.

Here we complete the proof of Proposition 4.3 and Proposition 4.4.

§ 5. The action of the Weyl group.

In this section we give the proof to the main part of our main theorems.

Let $X \subset \mathbb{P}^3$ be a normal quartic surface [resp. Let $\pi: X \longrightarrow \mathbb{P}^2$ be a branched double covering over $\mathbb{P}^2$ branching along a reduced sextic curve B.] with a singularity $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} at $x_0 \in X$. We assume that other singularities on X than $x_0 \in X$ are rational double points. Let $\rho: Z \longrightarrow X$ be the minimal resolution of singularities. Let $D = \rho^{-1}(x_0)$. Then for a suitably chosen α and ι, $\underline{Z} = (Z, D, \alpha, \iota)$ is a marked rational surface of degree -1. (Cf. Lemma 1.3, Proposition 1.4, Definition 2.4.) Moreover by exchanging α by αw with a suitable $w \in W_P$, we can assume that either $\alpha(\lambda_1) = L$ or $\alpha(\lambda_2) = L$ holds, where $\lambda_1 = 7\varepsilon_0 - 3\varepsilon_1 - 2\varepsilon_2 - \cdots -2\varepsilon_{10}$, $\lambda_2 = 9\varepsilon_0 - 3\varepsilon_1 - \cdots -3\varepsilon_8 - 2\varepsilon_9 - \varepsilon_{10}$ and $L = \rho^*\mathcal{O}_{\mathbb{P}^3}(1)$. (Cf. Proposition 4.3) [resp. we can assume that $\alpha(\lambda_3) = \rho^*\pi^*\mathcal{O}_{\mathbb{P}^3}(1) = L$ holds where $\lambda_3 = 6\varepsilon_0 - 2\varepsilon_1 - 2\varepsilon_2 - \cdots -2\varepsilon_8 - \varepsilon_9 - \varepsilon_{10}$. (Cf. Proposition 4.4)] Since the restriction of L to D is trivial, the characteristic homomorphism $\phi_{\underline{Z}}: \Gamma \longrightarrow E$ satisfies $\phi_{\underline{Z}}(\lambda_i) = 0$ and belongs to the subset $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_i, E)$ of $\mathrm{Hom}(\Gamma, E)$ where $i = 1$ or 2 according as $\alpha(\lambda_1) = L$ or $\alpha(\lambda_2) = L$. [resp. the characteristic homomorphism $\phi_{\underline{Z}}: \Gamma \longrightarrow E$ satisfies $\phi_{\underline{Z}}(\lambda_3) = 0$ and belongs to the subset $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_3, E)$ of $\mathrm{Hom}(\Gamma, E)$.] (Cf. Definition 2.6) Furthermore the kernel $\mathrm{Ker}\,\phi_{\underline{Z}}$ contains no element $\mu \in \Gamma$ with $\mu^2 = 0$ and $\mu \cdot \lambda_i = 2$. $(i = 1,2)$ (Cf. Theorem 3.25) [resp. the kernel $\mathrm{Ker}\,\phi_{\underline{Z}}$ contains no element $\mu \in \Gamma$ with $\mu^2 = 0$ and $\mu \cdot \lambda_3 = 1$. (Cf.

Theorem 3.28)]

Conversely for a fixed $i = 1$ or 2 choose an element $\phi \in \mathrm{Hom}(\Gamma, E)$ such that

(1) $\phi(\lambda_i) = 0$ and

(2) $\mathrm{Ker}\,\phi$ contains no element μ with $\mu^2 = 0$ and $\mu \cdot \lambda_i = 2$.

[resp. Conversely choose an element $\phi \in \mathrm{Hom}(\Gamma, E)$ such that

(1) $\phi(\lambda_3) = 0$ and

(2) $\mathrm{Ker}\,\phi$ contains no element μ with $\mu^2 = 0$ and $\mu \cdot \lambda_3 = 1$.]

Then by theorem 2.8 there exists a marked rational surface $\underline{Z} = (Z, D, \alpha, \iota)$ with $\phi = \phi_{\underline{Z}}$. Exchanging α by $w\alpha$ where $w \in W_S$ is an element of the Weyl group associated to nodal roots, we can assume that $\alpha(\lambda_i) \in V_S \cap C_+$ [resp. $\alpha(\lambda_3) \in V_S \cap C_+$] and $\phi = \phi_{\underline{Z}}$, since $V_S \cap C_+$ is a fundamental domain of W_S. By Proposition 4.1 and since it follows from the above condition that $L|_D \cong \mathcal{O}_D$ for $L = \alpha(\lambda_i) \in \mathrm{Pic}(Z)$ [resp. $L = \alpha(\lambda_3) \in \mathrm{Pic}(Z)$], the line bundle L is a polarization of Z. Moreover by the above condition (2) and by Theorem 3.25, L defines a morphism $\phi_L : Z \longrightarrow X \subset \mathbb{P}^3$ to a normal quartic surface [resp. Moreover by the above condition (2) and by Theorem 3.28, L defines a morphism $\Phi: Z \longrightarrow X \subset \mathbb{P}(1, 1, 1, 3)$ to a branched double covering over $\mathbb{P}^2$ branching along a reduced sextic curve B] with singularity $\tilde{E}_8$, $T_{2,3,7}$, or E_{12} according as E is an elliptic curve, $\mathbb{C}^*$ or $\mathbb{C}$.

Note that by Proposition 3.31, singularities on X are described by $\Pi \cap \mathrm{Ker}\,\phi_{\underline{Z}} \cap (\mathbb{Z}\lambda_i)^{\perp}$ $(i = 1, 2, 3)$ where Π is the set of roots in P and $(\mathbb{Z}\lambda_i)^{\perp}$ is the orthogonal complement of λ_i in

$\Gamma = (\mathbb{Z}\kappa)^{\perp}$.

Thus classification of singularities of surfaces under consideration is reduced to studying the abelian group $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_i, E)$. ($i = 1, 2, 3$)

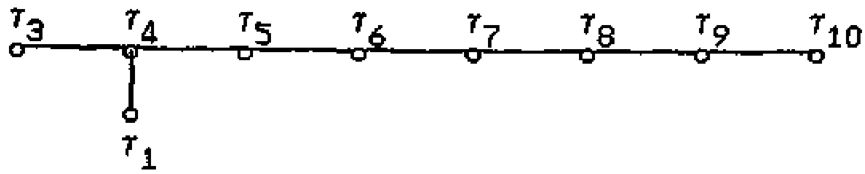
Let Λ be the orthogonal complement of $\mathbb{Z}\lambda_i$ in Γ. We define a homomorphism

$$u: \Gamma \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{Z}) = \Lambda^*$$

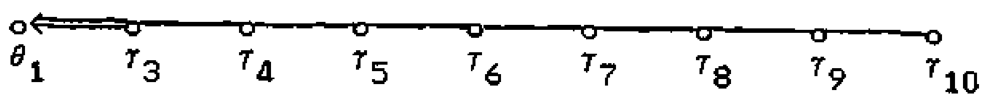
by $u(\alpha)(\xi) = \alpha\cdot\xi$ for $\alpha\in\Gamma$ and $\xi\in\Lambda$. It is easy to see that its kernel is $\Lambda^{\perp} = \mathbb{Z}\lambda_i$ and it is surjective since Γ is a unimodular lattice. Thus it induces an isomorphism $\bar{u}: \Gamma/\mathbb{Z}\lambda_i \xrightarrow{\sim} \Lambda^*$. In what follows we sometimes consider $\bar{\phi}\in\mathrm{Hom}(\Lambda^*, E)$ instead of $\phi\in\mathrm{Hom}(\Gamma, E)$ with $\phi(\lambda_i) = 0$. Since $\bar{u}$ is bijective they are equivalent. Note that the composition $\Lambda \longrightarrow \Gamma \longrightarrow \Gamma/\mathbb{Z}\lambda_i \xrightarrow{\sim} \Lambda^*$ is injective since $\Lambda \cap \mathbb{Z}\lambda_i = \{0\}$. We regard Λ as a subset of Λ^* by this injective mapping. Conversely Λ^* is regarded as a subset of $\Lambda\otimes\mathbb{Q}$. We can define a bilinear form on Λ^* with values in rational numbers by extending that on Λ. For any element $0 \neq \theta\in\Lambda\otimes\mathbb{Q}$, the refrection s_θ with respect to the hyperplane orthogonal to θ is defined by $s_\theta(x) = x - \frac{2(x\cdot\theta)}{(\theta\cdot\theta)}\theta$ for $x\in\Lambda\otimes\mathbb{Q}$. It is an automorphism of order 2 preserving the linear form. (In what follows an affine automorphism of order 2 of an affine space whose set of fixed points has codimension 1 is called a <u>reflection.</u>)

Now we would like to give a remark. Let A be an arbitrary abelian group. When a group G acts on Λ we define an action of G on $\mathrm{Hom}(\Lambda, A)$ by $(gF)(\xi) = F(g^{-1}(\xi))$ for $g\in G$, $F\in\mathrm{Hom}(\Lambda, A)$, and $\xi\in\Lambda$. With this definition the inclusion $\Lambda \longrightarrow \Lambda^*$ is an equivaliant homomorphism if the action preserves the bilinear form.

Next we consider the case concerning $\lambda_1 = 7\varepsilon_0-3\varepsilon_1-2\varepsilon_2-\cdots-2\varepsilon_{10}$. Set $\Xi_1 = \mathbb{Z}\tau_1+\mathbb{Z}\tau_3+\mathbb{Z}\tau_4+\mathbb{Z}\tau_5+\mathbb{Z}\tau_6+\mathbb{Z}\tau_7+\mathbb{Z}\tau_8+\mathbb{Z}\tau_9+\mathbb{Z}\tau_{10}$. (τ_2 does not appear.) It is easy to see that the orthogonal complement of $\mathbb{Z}\lambda_1$ in Γ is Ξ_1 (i.e., $\Lambda = \Xi_1$) and that Ξ_1 is the root lattice of type D_9.



Let W_{Ξ_1} be the group generated by s_{τ_1}, s_{τ_3}, , $s_{\tau_{10}}$. It is the Weyl group of type D_9. W_{Ξ_1} acts on Ξ_1 and Ξ_1^*. Set $\omega_1 = \frac{1}{4}\tau_1 - \frac{1}{4}\tau_3 + \frac{1}{2}\tau_4 + \frac{1}{2}\tau_6 + \frac{1}{2}\tau_8 + \frac{1}{2}\tau_{10}$. We can check that $\Xi_1^* = \Xi_1+\mathbb{Z}\omega_1$. Set $\theta_1 = \frac{1}{2}\tau_1 - \frac{1}{2}\tau_3$. One can see easily $\theta_1\in\Xi_1^*$ and $\theta_1^2 = -1$. Moreover $2\theta_1\cdot\Xi_1^* \subset \mathbb{Z}$ since $\theta_1\cdot\omega_1 = -\frac{1}{2}$ and $\theta_1\,\Xi_1 \subset \mathbb{Z}$. Note that it implies that the reflection $s_{\theta_1}(x) = x+2(x\cdot\theta_1)\theta_1$ defines a homomorphism Ξ_1^* to Ξ_1^*. Let G_1 be the subgroup of the orthogonal group of Ξ_1^* generated by s_{θ_1}, s_{τ_3}, s_{τ_4}, s_{τ_5}, s_{τ_6}, s_{τ_7}, s_{τ_8} and $s_{\tau_{10}}$. The group G_1 is the Weyl group of type B_9 since the mutual intersection numbers of θ_1, τ_3, , τ_{10} give the following Dynkin graph.



Lemma 5.1. Every element $\xi\in\Xi_1^*$ with $\xi^2 = -1$ is conjugate to θ_1 with respect to the action of G_1. Moreover every element

$\xi \in \Xi_1^*$ with $\xi^2 = -2$ is conjugate to τ_3 with respect to the action of W_{Ξ_1}.

Proof. We first show that every element $\xi \in \Xi_1^*$ with $\xi^2 = -1$ or $\xi^2 = -2$ belongs to the free submodule Γ' generated by θ_1, τ_3, τ_4, $\ldots$, τ_{10}. Otherwise we have an element $y \in \Gamma'$ with $x = y + \omega_1$ since $[\Xi_1^* : \Gamma'] = 2$. It is easily checked that the restriction of the intersection form to Γ' has values in $\mathbb{Z}$. Thus y^2 and $2y \cdot \omega_1$ are integers since $2\omega_1 \in \Gamma'$. It follows that $\omega_1{}^2 = \xi^2 - y^2 - 2y \cdot \omega_1$ is an integer. However we have $\omega_1{}^2 = -9/4$, a contradiction. Secondly we show that every element $\xi \in \Xi_1^*$ with $\xi^2 = -2$ belongs to Ξ_1. We may assume that $\xi \in \Gamma'$. Assume moreover that $\xi \notin \Xi_1$. Then we have an element $z \in \Xi_1$ with $\xi = z + \theta_1$ since $[\Gamma' : \Xi_1] = 2$. It follows that $\theta_1{}^2 = \xi^2 - z^2 - 2\theta_1 \cdot z$ is an even integer. However $\theta_1{}^2 = -1$, which is a contradiction. Since Γ' and Ξ_1 are the root lattices of type B_9 and D_9 respectively one obtains the desired claim by the theory of root systems. Q.E.D.

Corollary 5.2. (1) Every element $\tau \in \Xi_1 \subset \Gamma$ with $\tau^2 = -2$ is a root. (Recall that an element $\tau \in \Gamma$ conjugate to some τ_i $(1 \leq i \leq 10)$ with respect to W_P called a root.)

(2) For every element $\theta \in \Xi_1^*$ with $\theta^2 = -1$, the reflection s_θ belongs to G_1.

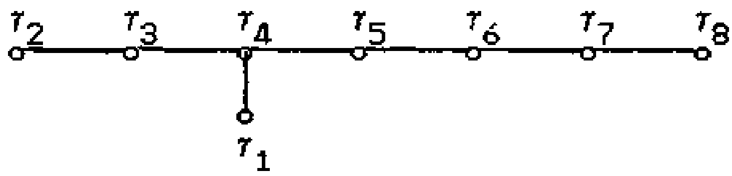
(3) For every element $\theta \in \Xi_1^*$ with $\theta_1{}^2 = -1$, we have an element $\xi \in \Xi_1^*$ with $2\xi \cdot \theta = 1$.

(4) For every element $\eta \in \Xi_1{}^*$ with $\eta^2 = -2$, we have an element $\xi \in \Xi_1{}^*$ with $\xi \cdot \eta = 1$.

Proof. (1) Since $G_1 \subset W_P$ it is obvious.
(2) There is $g \in G_1$ with $\theta = g(\theta_1)$. Thus $s_\theta = g s_{\theta_1} g^{-1} \in G_1$.
(3) Since $2(\omega_1+\tau_3)\cdot\theta_1 = 1$, $2g(\omega_1+\tau_3)\ \theta = 1$ for $\theta = g(\theta_1)$.
(4) We can assume that $\eta = g(\tau_3)$ for $g \in G_1$. Then $g(\tau_4)$ has the desired property. Q.E.D.

Let Π_1 be the set of all elements $\xi \in \Xi_1{}^*$ with $\xi^2 = -1$ or -2. Π_1 is the root system of type B_9. Ξ_1 is identified with the co-root lattice $Q(\Pi_1{}^\vee)$, i.e., the free module generated by co-roots. $\Xi_1{}^*$ is the weight lattice $P(\Pi_1)$. Moreover $\Gamma' = Q(\Pi_1) = P(\Pi_1{}^\vee)$.

Let us proceed to the case concerning to $\lambda_2 = 9\varepsilon_0-3\varepsilon_1-3\varepsilon_2-\cdot -3\varepsilon_8-2\varepsilon_9-\varepsilon_{10}$. Set $\Xi_2 = \mathbb{Z}\tau_1+\mathbb{Z}\tau_2+\mathbb{Z}\tau_3+\mathbb{Z}\tau_4+\mathbb{Z}\tau_5+\mathbb{Z}\tau_6+\mathbb{Z}\tau_7+\mathbb{Z}\tau_8$ and $\omega_2 = 3\varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7-\varepsilon_8-2\varepsilon_9+\varepsilon_{10}$. Ξ_2 is the root lattice of type E_8 and it is easy to see that the orthogonal complement Λ



of $\mathbb{Z}\lambda_2$ in Γ is the orthogonal direct sum of $\mathbb{Z}\omega_2$ and Ξ_2, i.e., $\Lambda = \mathbb{Z}\omega_2+\Xi_2$. Thus we have $\Lambda^* = \mathbb{Z}(\omega_2/4)+\Xi_2{}^*$. Let G_2' be the Weyl group of type E_8 generated by s_{τ_1}, s_{τ_2}, s_{τ_3}, s_{τ_4}, s_{τ_5}, s_{τ_6}, s_{τ_7} and s_{τ_8}. G_2' acts on $\mathbb{Z}\omega_2$ trivially. Let T be a cyclic group of order 2 generated by the reflection $s_{(\omega_2/2)}$ acting on $\Lambda^* = \mathbb{Z}(\omega_2/4)+\Xi_2^*$. T acts on Ξ_2^* trivially and acts on $\mathbb{Z}(\omega_2/4)$ as the change of the sign; $\alpha \longrightarrow -\alpha$. We set $G_2 = T\times G_2'$.

Lemma 5.3. (1) If $\theta^2 = -1$ for $\theta\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$, then $\theta = \pm\omega_2/2$.

(2) If $\eta^2 = -2$ for $\eta\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$, then $\eta\in\Xi_2^*$ and such an element η is conjugate to each other with respect to the action of G_2'.

Proof. (1) Set $\theta = (m\omega_2/4)+\xi$ with $m\in\mathbb{Z}$, $\xi\in\Xi_2^*$. We have $-1 = -(m^2/4)+\xi^2$ since $\omega_2^2 = -4$. Since ξ^2 is a negative integer unless $\xi = 0$, one sees that $m = \pm 2$ and $\xi = 0$.

(2) We set $\eta = (m\omega_2/4)+\xi$ with $m\in\mathbb{Z}$, $\xi\in\Xi_2^*$. We have $-2 = -(m^2/4)+\xi^2$. Thus $m = 0$ and $\eta\in\Xi_2^*$ since ξ^2 is a non-positive even integer and since $8 = 2\times 4$ is not a square of an integer. Every element $\eta\in\Xi_2^*$ with $\eta^2 = -2$ is conjugate with respect to G_2' since Ξ_2^* is the root lattice of type E_8. Q.E.D.

Corollary 5.4. (1) Every element $\gamma\in\mathbb{Z}(\omega_2/4)+\Xi_2^* \subset \Gamma$ with $\gamma^2 = -2$ is a root.

(2) For every element $\theta\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $\theta^2 = -1$, the reflection s_θ belongs to T.

(3) For every element $\theta\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $\theta^2 = -1$, we have an element $\xi\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $2\xi\ \theta = 1$.

(4) For every element $\eta\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $\eta^2=-2$ we have an element $\xi\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $\xi\cdot\eta=1$.

Let Π_2 be the set of elements $\xi\in\mathbb{Z}(\omega_2/4)+\Xi_2^*$ with $\xi^2=-1$ or -2. Π_2 is the root system of type A_1+E_8. The irreducible comoponent of type A_1 is consisted of $\{\pm\omega_2/2\}$ and they are regarded as short roots compared with those in the system of type E_8. Equalities $Q(\Pi_2^\vee)=\mathbb{Z}\omega_2+\Xi_2^*$, $Q(\Pi_2)=P(\Pi_2^\vee)=\mathbb{Z}(\omega_2/2)+\Xi_2^*$, $P(\Pi_2)=\mathbb{Z}(\omega_2/4)+\Xi_2^*$ holds.

Lemma 5.5. Assume $i=1$ or 2. Let Λ be the orthogonal complement of $\mathbb{Z}\lambda_i$ in Γ. The following conditions are equivalent for $\phi\in\mathrm{Hom}(\Lambda^*, E)$.

(a) There exists an element $\mu\in\Gamma$ with $\mu^2=0$, $\mu\cdot\lambda_i=2$ and $\phi u(\mu)=0$.

(b) There exists an element $\theta\in\Lambda^*$ with $\theta^2=-1$ and $\phi(\theta)=0$.

(c) There exists an element $\theta\in\Lambda^*$ with $\theta^2=-1$ such that $s_\theta(\phi)=\phi$.

Proof. (a)$\Longrightarrow$(b). Recall the definition of u. Since $\Gamma\subset\mathbb{Z}(\lambda_i/4)+\Lambda^*$, every element $\alpha\in\Gamma$ can be written uniquely as $\alpha=(m\lambda_i/4)+\alpha'$ with $m\in\mathbb{Z}$, $\alpha'\in\Lambda^*$. Then $\alpha'=u(\alpha)$. Thus set $\theta=u(\mu)$. We have $\mu=(\lambda_i/2)+\theta$ since $\mu\cdot\lambda_i=2$. We have $\theta^2=((\lambda_i/2)-\mu)^2=1-2+0=-1$ and $\phi(\theta)=\phi u(\mu)=0$.

(b)$\Longrightarrow$(a). Since u is surjective, there is an elemnt $\mu'\in\Gamma$

with $\theta = u(\mu')$. Then there is an integer $m \in \mathbf{Z}$ with $\mu' = (m\lambda_i/4)+\theta$. We have $(\mu')^2 = m^2/4-1$, which implies that $m = 4n+2$ for some integer n, since $(\mu')^2$ is an even integer. (Γ is an even lattice.) Set $\mu = \mu'-n\lambda_i$. Then $\mu \in \Gamma$, $\mu^2 = 0$, $\mu\cdot\lambda_i = 2$ and $\phi u(\mu) = 0$.

(b)$\Longrightarrow$(c). If (b) is satisfied, then for $x \in \Lambda^*$, $(s_\theta(\phi))(x) = \phi(s_\theta(x)) = \phi(x+2(x\cdot\theta)\theta) = \phi(x)+2(x\cdot\theta)\phi(\theta) = \phi(x)$.

(c)$\Longrightarrow$(b). Note that there is an element $\xi \in \Lambda^*$ with $2\xi\ \theta = 1$. (Corollary 5.2, Corollary 5.4.) If (c) is satisfied, then $\phi(\xi) = \phi s_\theta(\xi) = \phi(\xi)+\phi(\theta)$. Thus $\phi(\theta) = 0$. Q.E.D.

The above lemma implies that the criterion for whether the marked rational surface can be realized as a quartic surface or not can be interpreted with group-theoretic words.

To help reader's understanding we write down one more lemma.

Lemma 5.6. For every element $\gamma \in \Lambda$ with $\gamma^2 = -2$, the following conditions are equivalent.

(a) $\phi u(\gamma) = 0$.

(b) $\phi(\gamma) = 0$.

(c) $s_\gamma(\phi) = \phi$.

Proof. Here we only give the proof of (c)$\Longrightarrow$(b). The other parts are trivial. Recall that there is an element $\xi \in \Lambda^*$ with $\xi\cdot\gamma = 1$. (Corollary 5.2, Corollary 5.4) If (c) is satisfied, then

$\phi(\xi) = \phi s_\tau(\xi) = \phi(\xi)+\phi(\tau)$. Thus $\phi(\tau) = 0$. Q.E.D.

Summing up the above results we have the following proposition.

Proposition 5.7. Assume $i = 1$ or 2. Let Λ be the orthogonal complement of $\mathbb{Z}\lambda_i$ in Γ and $u: \Gamma \longrightarrow \Lambda^*$ be the canonical surjection. Let G_i be the group generated by all reflections s_η corresponding to elements $\eta \in \Lambda^*$ with $\eta^2 = -1$ or -2. The following conditions are equivalent for $\phi \in \mathrm{Hom}(\Lambda^*, E)$.

(A) There exists a marked rational surface $\underline{Z} = (Z, D, \alpha, \iota)$ over E of degree -1 such that

(i) the characteristic homomorphism $\phi_{\underline{Z}}$ of $\underline{Z}$ coincides with ϕu;

(ii) the line bundle $L = \alpha(\lambda_i)$ defines a generically one-to-one morphism $\phi_L: Z \longrightarrow X \subset \mathbb{P}^3$ to a normal quartic surface X; and

(iii)the configuration of singularities on X is a unique $\tilde{E}_8$, $T_{2,3,7}$, or E_{12}(It depends on whether E is an elliptic curve, $\mathbb{C}^*$, or $\mathbb{C}$.) plus a configuration of rational double points associated to the set of Dynkin graphs $\Sigma p_k A_k + \Sigma q_\ell D_\ell + \Sigma r_m E_m$.

(B) The kernel $\mathrm{Ker}\ \phi$ contains no element $\theta \in \Lambda^*$ with $\theta^2 = -1$ and the set of elements $\eta \in \Lambda^*$ with $\eta^2 = -2$, $\phi(\eta) = 0$ is the root system of type $\Sigma p_k A_k + \Sigma q_\ell D_\ell + \Sigma r_m E_m$.

(C) The isotropy group $I_{G_i}(\phi) = \{ g \in G_i \mid g(\phi) = \phi \}$ of ϕ with respect to G_i contains <u>no</u> reflections associated to any element

$\theta \in \Lambda^*$ with $\theta^2 = -1$ and moreover the maximal subgroup of $I_{G_i}(\phi)$ generated by reflections is the Weyl group of type $\Sigma p_k A_k + \Sigma q_\ell D_\ell + \Sigma r_m E_m$.

Remark. The group G_1 is the Weyl group of type B_9 and G_2 is the Weyl group of type A_1+E_8. In the latter case the irreducible component of type A_1 corresponds to the elements $\theta \in \Lambda^*$ with $\theta^2 = -1$.

Now our classification is reduced to the classification of subgroups of G_i which can be realized as the maximal subgroup generated by reflections of $I_{G_i}(\phi) = \{ g \in G \mid g(\phi) = \phi \}$ for some $\phi \in \mathrm{Hom}(\Lambda^*, E)$.

Definition 5.8. The following procedure which associates a root system R to its root subsystem R' is called the <u>elementary transformation</u> of the root system.

(1) We divide R into the direct sum of irreducible root system, say $R = \bigoplus_i R_i$.

(2) We choose a fundamental system of roots for every i, say $\Delta_i \subset R_i$.

(3) For every i, we choose a proper subset $\tilde{\Delta}_i$ of the union $\Delta_i \cup \{-\eta_i\}$ where η_i is the highest root associated to Δ_i.

(4) We set $R' = \bigoplus_i R'_i$ where R'_i is the root system generated by $\tilde{\Delta}_i$.

Proposition 5.9. When E is an irreducible smooth elliptic curve (resp. $\mathbb{C}^*$), the following conditions are equivalent for any subgroup H of the Weyl group $W = W(R)$ associated to a fixed root system R. We denote by Q the co-root lattice of R, i.e., the free $\mathbb{Z}$-module generated by co-roots $\{ \eta^\vee \mid \eta\in R \}$.

(1) The group H coincides with the maximal subgroup generated by reflections of the isotropy group $I_W(\phi)$ for some $\phi\in Q\otimes E$.

(2) The group H is generated by a set of reflections $\{ s_\eta \mid \eta\in R'\}$ where R' is a root subsystem of R which is obtained by elementary transformations repeated <u>twice</u> (resp. <u>only once</u>.) from R.

Proof. Let $\hat{Q}$ be the root lattice of R. The vector space $Q\otimes\mathbb{R}$ is regarded as the dual space of $\hat{Q}\otimes\mathbb{R}$. We denote the canonical pairing $Q\otimes\mathbb{R} \times \hat{Q}\otimes\mathbb{R} \longrightarrow \mathbb{R}$ by $\langle\ ,\ \rangle$.

We first assume that E is an elliptic curve. We have representation $E = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ where $\tau\in\mathbb{C}$ and $\operatorname{Im}\tau > 0$. We fix such representation. The covering mapping $\pi\colon \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ induces the covering mapping $\bar{\pi}\colon Q\otimes\mathbb{C} \longrightarrow Q\otimes E$. Set $\bar{W} = W \ltimes (Q\oplus Q)$ where $\ltimes$ denotes the semi-direct product with respect to the diagonal action of W to $Q\oplus Q$. (i.e., for $g\in W$, $(\xi', \xi'')\in Q\oplus Q$, $g(\xi', \xi'') = (g\xi', g\xi'')$.) The group $\bar{W}$ acts on $Q\otimes\mathbb{C}$ by $(g, \xi', \xi'')(\phi'+\tau\phi'') = (g(\phi')+\xi')+\tau(g(\phi'')+\xi'')$ where $g\in W$, $\xi', \xi''\in Q$ and $\phi', \phi''\in Q\otimes\mathbb{R}$. We have a canonical isomorphism of isotropy groups. $I_{\bar{W}}(\bar{\phi}) \cong I_W(\bar{\pi}(\bar{\phi}))$

for $\bar{\Phi}\in Q\otimes\mathbb{C}$. Thus we can consider the action of $\bar{W}$ on $Q\otimes\mathbb{C}$ instead of that of W on $Q\otimes E$.

Set $W_a = W \ltimes Q$. The group W_a is the affine Weyl group of R. We have a diagram

$$\begin{tikzcd} \bar{W} \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & W_a \arrow[d, "\nu_2"] \\ W_a \arrow[r, "\nu_1"'] & W \end{tikzcd}$$

where $\rho_1(g, \xi', \xi") = (g, \xi')$, $\rho_2(g, \xi', \xi") = (g, \xi")$ and $\nu_i(g, \xi') = g$ $(i = 1, 2)$. Set $\bar{\Phi} = \Phi'+\tau\Phi"$ with $\Phi', \Phi"\in Q\otimes\mathbb{R}$. Let $(g, \xi')\in I_{W_a}(\Phi')$. We have $g(\Phi')+\xi' = \Phi'$ and one sees that ξ' is uniquely determined by g and Φ'. Thus the restriction $\nu_1|I_{W_a}(\Phi')$ of ν_1 is injective. Set $J(\Phi') = \nu_1(I_{W_a}(\Phi'))$. $J(\Phi')$ is isomorphic to $I_{W_a}(\Phi')$ and $\nu_2^{-1}J(\Phi') = J(\Phi') \ltimes Q$ is isomorphic to $\rho_1^{-1}I_{W_a}(\Phi')$ via ρ_2. We have

(5.1) $\quad I_{\bar{W}}(\bar{\Phi}) = \rho_1^{-1}I_{W_a}(\Phi') \cap \rho_2^{-1}(\Phi") \cong I_{J(\Phi')} \ltimes Q^{(\Phi")}$.

We claim here that there is a root subsystem R' of R which is obtained from R and $J(\Phi')$ is the Weyl group generated by $\{ s_\eta \mid \eta\in R' \}$ and that conversely for any root subsystem R' obtained by one elementary transformation from R, there is a point $\Phi'\in Q\otimes\mathbb{R}$ such that $J(\Phi')$ coincides with the Weyl group generated by $\{ s_\eta \mid \eta\in R' \}$.

To see this recall that the action of W_a on $Q\otimes\mathbb{R}$ has a fundamental domain C_0. C_0 is called a small Weyl chamber. (Cf. Bourbaki [3]) Since every small Weyl chamber is conjugate we can assume that $\phi \in \bar{C}_0$. ($\bar{}$ denotes the closure.) Now let s_H denote the reflection of $Q\otimes\mathbb{R}$ in W_a whose set of fixed points coincides with a hyperplane H. Let $\underline{M}$ be the set of all hyperplanes H with $s_H \in W_a$. The domain C_0 is a connected component of $Q\otimes\mathbb{R} - \bigcup_{H\in\underline{M}} H$. Set $\underline{M}_0 = \{ H\in\underline{M} \mid \dim (H \cap \bar{C}_0) = \dim H \}$. $\underline{M}_0$ is the set of walls of the small chamber C_0. It is known that for every $H\in\underline{M}_0$ there is a unique root $\eta\in R$ perpendicular to H and such that $\langle x, \eta\rangle > 0$ for $x\in C_0$. We denote it by $\eta(H)$. Let $R = \bigoplus_i R_i$ be the decomposition into irreducible root systems. Then there is a fundamental system of roots $\Delta_i \subset R_i$ for each i such that the union $\bigcup_i \Delta_i \cup \{-\eta_i\}$ coincides with the set $\{ \eta(H) \mid H\in\underline{M}_0 \}$ where η_i is the highest root of R_i associated to Δ_i. Let $\underline{M}_0(\phi') = \{ H\in\underline{M}_0 \mid \phi'\in H \}$. It is the set of walls of C_0 passing through ϕ'. Then it is also known that the isotropy group $I_{W_a}(\phi')$ coincides with the subgroup of W_a generated by $\{ s_H \mid H\in\underline{M}_0(\phi') \}$, the set of reflections corresponding to walls of C_0 passing through ϕ. Since the intersection of all walls of the small Weyl chamber of an irreducible root system is empty, for every i, $(\Delta_i \cup \{-\eta_i\}) \cap \{ \eta(H) \mid H\in\underline{M}_0(\phi') \}$ is a proper subset of $\Delta_i \cup \{-\eta_i\}$. Let R' be the root system generated by $\{ \eta(H) \mid H\in\underline{M}_0(\phi') \}$, the set of roots perpendicular to some wall of C_0 passing through ϕ' and directed to the inside of C_0. By the construction R' is the one obtained by one elementary transformation from R and $J(\phi')$ is the Weyl group generated by $\{ s_\eta \mid \eta\in R' \}$.

Conversely let R' be a root subsystem of $R = \bigoplus_i R_i$ obtained by one elementary transformation from R. Choosing the fundamental system of $\Delta_i \subset R_i$ of the irreducible root system R_i is equal to choosing a Weyl chamber C_i of $W(R_i)$ in $Q_i \otimes \mathbb{R}$ where Q_i is the co-root lattice of R_i. Let C_{i0} be the small Weyl chamber contained in C_i and such that $0 \in \bar{C}_{i0}$, which is the fundamental domain of $W_a(R_i) = W(R_i) \ltimes Q_i$. Let $\underline{M}_{i0} = \{ H\text{: hyperplane in } Q_i \otimes \mathbb{R} \mid s_H \in W_a(R_i),\ \dim (H \cap \bar{C}_{i0}) = \dim H \}$. $\underline{M}_{i0}$ is the set of walls of C_{i0}. Then the set $\{ \eta(H) \mid H \in \underline{M}_{i0} \}$ coincides with $\Delta_i \cup \{-\eta_i\}$ where η_i is the highest root. For the specified proper subset $\tilde{\Delta}_i$ of $\Delta_i \cup \{-\eta_i\}$ let ϕ_i' be a general point in the intersection $\bigcap \{ H \mid H \in \underline{M}_{i0},\ \eta(H) \in \tilde{\Delta}_i \}$. The isotropy group $I_{W_a(R_i)}(\phi_i')$ coincides with the Weyl group generated by $\{ s_\eta \mid \eta \in R_i' \}$ where R_i' is the root system generated by $\tilde{\Delta}_i$. Let ϕ' be the image of $\oplus \phi_i'$ by the inclusion $\oplus Q_i \otimes \mathbb{R} \subset Q \otimes \mathbb{R}$. One knows that the isotropy group $I_{W_a}(\phi')$ is the Weyl group generated by $\{ s_\eta \mid \eta \in \bigoplus_i R_i' = R' \}$. Thus we have the above claim.

In what follows we assume that $\phi' \in Q \otimes \mathbb{R}$ and R' has the relation mentioned in the above claim.

Let Q' be the co-root lattice associated to R'. Then $J(\phi') \ltimes Q'$ is the affine Weyl group associated to R'. Thus applying the above claim to R' one sees that subgroups H of W with the property (2) in Proposition 5.9 coincide with subgroups which can be written as $I_{J(\phi') \ltimes Q'}(\phi'')$ for some ϕ', $\phi'' \in Q \otimes \mathbb{R}$. Therefore by the equality (5.1) and by the next lemma we conclude that (1) and (2) are equivalent when E is an elliptic curve.

Lemma 5.10. Any reflection in $I_{J(\phi')} \ltimes Q^{(\phi^{\cdot})}$ belongs to $I_{J(\phi')} \ltimes Q'^{(\phi^{\cdot})}$. (Note that in general $Q \supsetneq Q'$.)

Proof. Any reflection in $W \ltimes Q$ can be written as (s_η, ξ) where $\eta \in R$ and $\xi \in Q$. Assume $(s_\eta, \xi) \in I_{J(\phi')} \ltimes Q^{(\phi^{\cdot})}$. We have $\eta \in R'$ and $\phi^{\cdot} - \langle \eta, \phi^{\cdot} \rangle \eta^{\vee} + \xi = \phi^{\cdot}$. Thus $\xi = \langle \eta, \phi^{\cdot} \rangle \eta^{\vee}$. Note that we have an element $w \in P(R)$ such that $\langle w, \eta^{\vee} \rangle = 1$. One sees that $\langle w, \xi \rangle = \langle \eta, \phi' \rangle$ is an integer since $P(R)$ is the dual lattice of Q. Thus we have $\xi \in Q'$ and $(s_\eta, \xi) \in J(\phi') \ltimes Q'$. Q.E.D.

Next assume $E = \mathbb{C}^*$. Let $\pi: \mathbb{C} \longrightarrow \mathbb{C}^*$ be the covering mapping. It induces the covering mapping $\bar{\pi}: Q \otimes \mathbb{C} \longrightarrow Q \otimes \mathbb{C}^*$. If $\bar{\pi}(\bar{\phi}) = \phi$ then $I_{W_a}(\bar{\phi}) \cong I_W(\phi)$, where $W_a = W \ltimes Q$. Thus the problem is reduced to the classification of isotropy groups of the action by W_a to $Q \otimes \mathbb{C}$. However note that the answer never changes by replacing $\mathbb{C}$ by $\mathbb{R}$ since the condition $g(\bar{\phi}) = \bar{\phi}$ for $g \in W_a$, $\bar{\phi} \in Q \otimes \mathbb{C}$ is written with an affine equation whose coefficients are all real numbers.

Pick $\chi \in Q \otimes \mathbb{R}$. Let C_0 be a small Weyl chamber whose closure contains χ. Then as mentioned above, $I_{W_a}(\chi)$ is the Weyl group generated by reflections associated to walls of C_0 passing through χ and moreover the set of generating reflections corresponds to a root system R' which is obtained by one elementary transformation from R.

We conclude the proof of both cases in Proposition 5.9.

Q.E.D.

Proposition 5.11. Let $W = W(R)$ be the Weyl group associated to a fixed root system R. Let Q be the co-root lattice of R. Then for any subgroup $H \subset W$, the following conditions are equivalent.

(1) For some $\phi \in Q \otimes \mathbb{C}$, $H = I_W(\phi)$.

(2) For some fundamental system of roots $\Delta \subset R$ and for some subset $\Delta' \subset \Delta$, H is the Weyl group generated by $\{ s_\eta \mid \eta \in R' \}$ where R' is the root system generated by Δ'.

Proof. For $g \in W$ and $\phi \in Q \otimes \mathbb{C}$, the condition $g(\phi) = \phi$ is described by a linear equation whose coefficients are all real numbers. Therefore we can replace $\mathbb{C}$ by $\mathbb{R}$. Pick $\chi \in Q \otimes \mathbb{R}$. Let C be the Weyl chamber of W such that the closure of C contains χ. Let $\underline{M}$ be the set of hyperplanes $H \subset Q \otimes \mathbb{R}$ such that for some reflection in W its fixed-point-set equals to H. A connected component of $Q \otimes \mathbb{R} - \bigcup_{H \in \underline{M}} H$ is C. Let $\underline{M}_0$ be the set of walls of C, i.e., $\underline{M}_0 = \{ H \in \underline{M} \mid \dim H = \dim (H \cap \bar{C}) \}$. For $H \in \underline{M}_0$ we have a unique root $\eta \in R$ perpendicular to H and $\langle x, \eta \rangle > 0$ for $x \in C$. If we denote it by $\eta(H)$, the set $\{ \eta(H) \mid H \in \underline{M}_0 \}$ is a fundamental system of roots of R. Moreover it is known that choosing a Weyl chamber C is equivalent to choosing a fundamental system of roots. Set $\Delta' = \{ \eta(H) \mid H \in \underline{M}_0, \chi \in H \}$. Δ' is the set of walls passing through χ. It is also known that $I_W(\chi)$ is the Weyl group generated by reflections $\{ s_\eta \mid \eta \in R' \}$, where R' is the root system generated by

Δ'. Thus (1) and (2) are equivalent. Q.E.D.

Now by Proposition 5.7, Remark just after Proposition 5.7, Proposition 5.9 and Proposition 5.11, the main parts of Theorem 0.2, Theorem 0.3 and Theorem 0.4 are obvious.

Recall that the intersection numbers of elements in the union of a fundamental system of an irreducible root system Δ and (−1) times its associated highest root are described by the extended Dynkin graph. Thus the elementary transformation of root systems corresponds to the elementary transformation of the Dynkin graphs. The series (I) in Theorem 0.2, Theorem 0.3 and Theorem 0.4 corresponds to $\lambda_1 = 7\varepsilon_0 - \quad -2\varepsilon_{10}$ and the series (II) corresponds to $\lambda_2 = 9\varepsilon_0 - \quad - \varepsilon_{10}$. However we did not necessarily use the expression containing B_9 or A_1+E_8 in those theorems. We used a simpler expression to say the same contents.

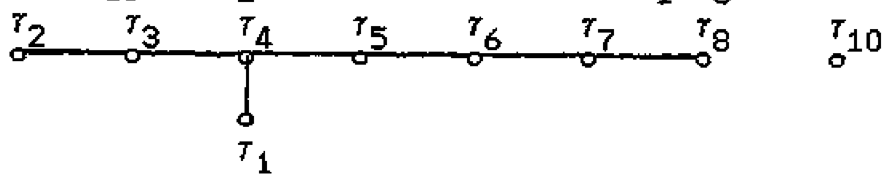
The part left unproved is the following proposition.

Proposition 5.12. (Umezu [21]) Assume that a normal quartic surface X has singularity $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} and that $\sum_{x\in X} p_g(X,x) \geqq 2$. Then X has only 2 singular points and both of them are of type $\tilde{E}_8$. Conversely a normal quartic surface with 2 singular points of type $\tilde{E}_8$ exists.

However this is Y. Umezu's result.

Let us proceed further to the case of branched double coverings.

In this case it is obvious that the orthogonal complement Λ of $\mathbb{Z}\lambda_3$ is the orthogonal direct sum of $\mathbb{Z}\tau_{10}$ and $\Xi_2 = \mathbb{Z}\tau_1+\mathbb{Z}\tau_2+\mathbb{Z}\tau_3+\mathbb{Z}\tau_4+\mathbb{Z}\tau_5+\mathbb{Z}\tau_6+\mathbb{Z}\tau_7+\mathbb{Z}\tau_8$. ($\lambda_3 = 6\varepsilon_0-2\varepsilon_1-2\varepsilon_2-2\varepsilon_3-2\varepsilon_4-2\varepsilon_5-2\varepsilon_6-2\varepsilon_7-2\varepsilon_8-\varepsilon_9-\varepsilon_{10}$.) Ξ_2 is the root lattice of type E_8. Let Π_3 be the set of all elements $\xi\in\mathbb{Z}\tau_{10}+\Xi_2$ with $\xi^2 = -2$. Π_3 is the root system of type A_1+E_8. The lattice $\mathbb{Z}\tau_{10}+\Xi_2$ is its root lattice and $\mathbb{Z}(\tau_{10}/2)+\Xi_2$ is its weight lattice. Moreover we have that $Q(\Pi_3) = Q(\Pi_3^{\vee}) = \mathbb{Z}\tau_{10}+\Xi_2$ and $P(\Pi_3) = P(\Pi_3^{\vee}) = \mathbb{Z}(\tau_{10}/2)+\Xi_2 = \Lambda^*$. Thus $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_3,\ E)$ is identified with $\mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+\Xi_2,\ E)$. We denote by G_3 the Weyl group generated by s_{τ_1}, s_{τ_2}, s_{τ_3}, s_{τ_4}, s_{τ_5}, s_{τ_6}, s_{τ_7}, s_{τ_8}, $s_{\tau_{10}}$. (s_{τ_9} does not appear.) The group G_3 acts on $\mathbb{Z}\tau_{10}+\Xi_2$ and $\mathbb{Z}(\tau_{10}/2)+\Xi_2$ and it is of type A_1+E_8.



The next lemma is easily checked.

Lemma 5.13. (1) Every elemnt $\tau\in\mathbb{Z}\tau_{10}+\Xi_2$ with $\tau^2 = -2$ is a root.

(2) For every $\tau\in\mathbb{Z}(\tau_{10}/2)+\Xi_2$ with $\tau^2 = -2$, we have $\xi\in\mathbb{Z}(\tau_{10}/2)+\Xi_2$ with $\tau\cdot\xi = 1$.

Thus Lemma 5.6 holds even when $i = 3$.

Lemma 5.14. The following conditions are equivalent for

$\phi \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+E_2, E)$.

(a) There exists an element $\mu \in \Gamma$ with $\mu^2 = 0$, $\mu \cdot \lambda_3 = 1$ and $\phi u(\mu) = 0$.

(b) $\pi_1(\phi) = 0$ where $\pi_1 : \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+E_2, E) \longrightarrow \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2), E)$ is the projection.

Proof. Let $\mu \in \Gamma$ be an element with $\mu^2 = 0$ and $\mu \cdot \lambda_3 = 1$. Since $\Gamma \subset \mathbb{Z}(\lambda_3/2)+\mathbb{Z}(\tau_{10}/2)+E_2$, we have an integer m and $\xi \in E_2$ such that $\mu = (\lambda_3/2)+(m\tau_{10}/2)+\xi$. (The coefficient of λ_3 is 1/2 since $\mu \cdot \lambda_3 = 1$.) It yields the equality $0 = \mu^2 = (1/2)-(m^2/2) +\xi^2$. Thus $m = \pm 1$ and $\xi = 0$ since ξ^2 is a negative integer unless $\xi = 0$. One knows $\mu = (\lambda_3/2)\pm(\tau_{10}/2)$. Since $u(\mu) = \pm\tau_{10}/2$, we have the desired equivalence. Q.E.D.

We have the following proposition.

Proposition 5.15. The following conditions are equivalent for $\phi \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+E_2, E)$. Let $\pi_1 : \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+E_2, E) \longrightarrow \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2), E)$ be the projection and G_3 be the Weyl group of the root lattice $\mathbb{Z}\tau_{10}+E_2$. (G_3 be of type A_1+E_8.)

(A) There exists a marked rational surface $\underline{Z} = (Z, D, \alpha, \iota)$ over E of degree -1 such that

(i) the characteristic homomorphism $\phi_{\underline{Z}}$ of $\underline{Z}$ coincides with ϕu;

(ii) the line bundle $L = \alpha(\lambda_3)$ defines a generically one-to-one morphism $\Phi: Z \longrightarrow X \subset \mathbb{P}(1,1,1,3)$ to a branched double covering over $\mathbb{P}^2$ branching along a reduced sextic curve B; and

(iii)the configuration of singularities on X is a unique $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} (It depends on whether E is an elliptic curve, $\mathbb{C}^*$ or $\mathbb{C}$.) plus a configuration of rational double points associated to the set of Dynkin graphs $\Sigma\, p_kA_k + \Sigma\, q_\ell D_\ell + \Sigma\, r_mE_m$.

(B) $\pi_1(\phi) \neq 0$ and the set of elements $\eta \in \mathbb{Z}(\tau_{10}/2)+\Xi_2$ satisfying $\eta^2 = -2$ and $\phi(\eta) = 0$ is the root system of type $\Sigma\, p_kA_k + \Sigma\, q_\ell D_\ell + \Sigma\, r_mE_m$.

(C) $\pi_1(\phi) \neq 0$ and the maximal subgroup generated by reflections of the isotropy group $I_{G_3}(\phi)$ is the Weyl group of type $\Sigma\, p_kA_k + \Sigma\, q_\ell D_\ell + \Sigma\, r_mE_m$.

Corollary 5.16. (1) Assume that E is an elliptic curve or $\mathbb{C}^*$. If $\pi_1(\phi) = 0$ for $\phi \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+\Xi_2,\ E)$, then we have another element $\phi' \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+\Xi_2,\ E)$ such that $\pi_1(\phi') \neq 0$ and $I_{G_3}(\phi') = I_{G_3}(\phi)$.

(2) Assume $E = \mathbb{C}$. Let G_3' be the subgroup of G_3 generated by s_{τ_1}, s_{τ_2}, $\ldots$, s_{τ_8}. If $\pi_1(\phi) \neq 0$ for $\phi \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+\Xi_2,\ \mathbb{C})$, then $I_{G_3}(\phi) = I_{G_3'}(\phi)$.

Proof. Let T be the cyclic group of order 2 generated by $s_{\tau_{10}}$ and π_2: $\mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+\Xi_2,\ E) \longrightarrow \mathrm{Hom}(\Xi_2,\ E)$ be the projection. Note that the equality $I_{G_3}(\phi) = I_T(\pi_1(\phi)) \times I_{G_3'}(\pi_2(\phi))$ holds.

(1) Let $\chi \in \mathrm{Hom}(\mathbb{Z}(\tau_{10}/2)+B_2, E)$ be the element with $\chi(B_2) = 0$, $\chi(\tau_{10}) = 0$ and $\chi(\tau_{10}/2) \neq 0$. If E is an elliptic curve or $\mathbb{C}^*$, such χ exists. The element $\phi' = \phi+\chi$ satisfies the above condition.

(2) If $E = \mathbb{C}$, then the condition $\chi(\tau_{10}) = 0$ and $\chi(\tau_{10}/2) = 0$ are equivalent. Thus if $\pi_1(\phi) \neq 0$, then $I_T(\pi_1(\phi))$ is the trivial group.

The important parts of Theorem 0.5, Theorem 0.6 and Theorem 0.7 follow from Proposition 5.15, Corollary 5.16, Proposition 5.9 and Proposition 5.11.

The parts left unproved are disconnectedness of strata in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$ and the case $\sum p_g(X,x) \geqq 2$. The case $\sum p_g(X,x) \geqq 2$ is treated in the last section.

The basis of disconnectedness is the following fact.

Fact 5.17. (Cf. Dynkin [6]) The root system R of type E_8 with the action of the Weyl group $W(R)$ contains two non-conjugate root subsystems of the following types.

(1) A_7 (2) $2A_3$ (3) A_5+A_1 (4) A_3+2A_1 (5) $4A_1$

Moreover both of non-conjugate ones of each type can be obtained by elementary transformations repeated twice from R.

According to this fact one knows for 10 cases in Theorem 0.5, (ii) there are two root subsystem R_1, R_2 of Π_3 of the same type such that for any automorphism of lattices $\beta\colon P \longrightarrow P$ satisfying $\beta(\kappa) = \kappa$ and $\beta(\lambda_3) = \lambda_3$, $\beta(R_1)$ never coincides with R_2. Indeed if we have a homomorphism β with $\beta(R_1) = R_2$, then $\beta(R_1 \cap \Xi_2) = R_2 \cap \Xi_2$ since the root subsystem $\Pi_3 \cap \Xi_2$ of Π_3 is the unique one of type E_8. However for type E_8 the Weyl group coincides with the automorphism group. Thus $R_1 \cap \Xi_2$ and $R_2 \cap \Xi_2$ are conjugate with respect to $W(\Xi_2 \cap \Pi_3)$.

Let E be a fixed elliptic curve. By Proposition 5.15, there are two marked rational surface of degree -1 over E, $\underline{Z}_1 = (Z_1, D_1, \alpha_1, \iota_1)$ and $\underline{Z}_2 = (Z_2, D_2, \alpha_2, \iota_2)$ such that $L_i = \alpha_i(\lambda_3)$ defines a morphism $\Phi_i\colon Z_i \longrightarrow X_i$ to a branched double covering $\pi_i\colon X_i \longrightarrow \mathbb{P}^2$ and $\operatorname{Ker} \phi_{\underline{Z}_i} \cap \Pi_3 = R_i$ $(i = 1,2)$. Thus for any intersection preserving homomorphism $\beta\colon \operatorname{Pic}(Z_1) \longrightarrow \operatorname{Pic}(Z_2)$ satisfying $\beta(\omega_{Z_1}) = \omega_{Z_2}$ and $\beta(\alpha_1(\lambda_3)) = \alpha_2(\lambda_3)$, two root subsystems $\beta(\operatorname{Ker}(\operatorname{Pic}(Z_1) \longrightarrow \operatorname{Pic}(D_1))) \cap \alpha_2(\Pi_3)$ and $\operatorname{Ker}(\operatorname{Pic}(Z_2) \longrightarrow \operatorname{Pic}(D_2)) \cap \alpha_2(\Pi_3)$ never coincide. However if the set of sextic curves with a configuration of singularities under consideration is connected, we get a contradiction by the following lemma.

Lemma 5.18. Let $\underline{B} \subset U \times \mathbb{P}^2$ be a family of reduced sextic curves over a connected analytic variety U, i.e., a subvariety of codimension 1 of $U \times \mathbb{P}^2$ such that for every $t \in U$, $B_t = \underline{B} \cap \{t\} \times \mathbb{P}^2$ is a reduced sextic plane curve. We assume that B_t has a unique $\tilde{E}_8$ singular point and other several rational singular points. We assume moreover that the number of each type of rational singular points is independent of $t \in U$. Let t' and t'' be arbitrary points on U. We define varieties X', X'', Z', Z'', D', D'' and morphisms π', π'', ρ', ρ'' as follows. The branched double coverings over $\mathbb{P}^2$ with the branch locus $B' = B_{t'}$ and $B'' = B_{t''}$ are $\pi'\colon X' \longrightarrow \mathbb{P}^2$ and $\pi''\colon X'' \longrightarrow \mathbb{P}^2$ respectively. The minimal resolution of singularities are denoted by $\rho'\colon Z' \longrightarrow X'$ and $\rho''\colon Z'' \longrightarrow X''$. Let D' and D'' be the exceptional curves of the simple elliptic singularities in X' and X'' respectively. We set $\Pi = \{ M \in \mathrm{Pic}(Z'') \mid M^2 = -2,\ M \cdot \omega_{Z''} = 0,\ M \cdot \rho''^* \pi''^* \mathcal{O}_{\mathbb{P}^2}(1) = 0 \}$. Then there is an intersection-form-preserving homomorphism $\beta\colon \mathrm{Pic}(Z') \longrightarrow \mathrm{Pic}(Z'')$ satisfying $\beta(\omega_{Z'}) = \omega_{Z''}$, $\beta(\rho'^* \pi'^* \mathcal{O}_{\mathbb{P}^2}(1)) = \rho''^* \pi''^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $\Pi \cap \beta(\mathrm{Ker}(\mathrm{Pic}(Z') \longrightarrow \mathrm{Pic}(D'))) = \Pi \cap \mathrm{Ker}(\mathrm{Pic}(Z'') \longrightarrow \mathrm{Pic}(D''))$.

Proof. If U is connected, we can choose finite points $t_1, t_2, \ldots, t_q \in U$ with $t' = t_1$, $t'' = t_q$ and analytic morphisms $f_i\colon T \longrightarrow U$, $1 \leq i < q$ from the unit disc $T = \{ z \in \mathbb{C} \mid |z| < 1 \}$ such that t_i and t_{i+1} belong to the image $f_i(T)$. Considering the pull-back of the family $\underline{B}$ by f_i instead of $\underline{B}$ itself, we can assume that U is the unit disc T without loss of generality.

Let $X_t \subset \mathbb{P}(1,1,1,3)$ be the branched double covering along $B_t \subset \mathbb{P}^2$. Obviously the set $\underline{X} = \bigcup_{t \in T} \{t\} \times X_t \subset T \times \mathbb{P}(1,1,1,3)$ is an analytic variety. Let Z_t be the minimal resolution of singularities of X_t. The set $\underline{Z} = \bigcup_{t \in T} \{t\} \times Z_t$ also has the structure of analytic variety. The relative Picard group $\mathrm{Pic}_{\underline{Z}/T}$ is a constant sheaf over T of free $\mathbb{Z}$-modules equipped bilinear forms. Let $\alpha\colon P_T \longrightarrow$

$\underline{Pic}_{\underline{Z}/T}$ be an isomorphis from the constant sheaf with values in P. Let β be the composition

$$\mathrm{Pic}(Z') = \mathrm{Pic}(Z_{t'}) \xleftarrow{\sim} (\underline{\mathrm{Pic}}_{\underline{Z}/T})_{t'} \xrightarrow{\alpha_{t'}} P \xleftarrow{\alpha_{t^{\bullet}}} (\underline{\mathrm{Pic}}_{\underline{Z}/T})_{t^{\bullet}} \xrightarrow{\sim} \mathrm{Pic}(Z_{t^{\bullet}}) = \mathrm{Pic}(Z').$$

Note that for any $\eta \in \mathrm{Pic}(Z_t)$ with $\eta^2 = -2$ such that η is orthogonal to the dualizing sheaf and the polarization, either η or $-\eta$ is effective if and only if η or $-\eta$ is the class of a exceptional divisor of the resolution of $Z_t \longrightarrow X_t$. By assumption that the configuration of singularities on B_t and thus on X_t is independent of $t \in T$, one sees that the above β has the desired property. Q.E.D.

§ 6. The case of ruled surfaces.

Let $\pi: X \longrightarrow \mathbb{P}^2$ be the branched covering branching along a reduced sextic curve B. Assume $P = \sum_{x \in X} p_g(X,x) \geq 2$. Under this assumption we study the structure of X in this section.

We owe ideas in this section greatly to Umezu [21].

Let $\rho: Z \longrightarrow X$ be the minimal resolution of singularities on X and $\sigma: Z \longrightarrow \bar{Z}$ be a morphism to a relatively minimal model. By Proposition 1.4, $\bar{Z}$ is a ruled surface over a smooth irreducible curve G of genus $P-1$. Let $p: \bar{Z} \longrightarrow G$ be the projection.

Let L be a general line in $\mathbb{P}^2$. Since L intersects with B at 6 points, the inverse image $\pi^{-1}(L)$ is a smooth curve of genus 2. Set $H = \rho^{-1}\pi^{-1}(L)$, which is also a smooth curve of genus 2.

Lemma 6.1. $P \leq 3$. Moreover if $P = 3$, then $\sigma(H)$ is a smooth curve of genus 2 and $p|\sigma(H): \sigma(H) \longrightarrow G$ is an isomorphism.

Proof. By the Hurwits formula for $p\sigma|_H: H \longrightarrow G$ we have $2 \geq m(2(P-1)-2)$ for some positive integer m. Thus $P \leq 3$. If $P = 3$, then $m = 1$ and the equality holds. It implies that $p\sigma$ is an unramified morphism of degree 1. Thus $\sigma|H$ and $p|\sigma(H)$ are isomorphisms.
Q.E.D.

We decompose σ into a composition of blowing-ups of points.

$$(6.1) \quad Z = Z_0 \xrightarrow{\sigma_0} Z_1 \xrightarrow{\sigma_1} Z_2 \xrightarrow{\sigma_2} \qquad \longrightarrow Z_{k-1} \xrightarrow{\sigma_{k-1}} Z_k = \bar{Z}$$

where σ_i is the blowing-up of a point $z_i \in Z_{i+1}$. Note that Z has an anti-canonical effective divisor D by Lemma 1.3. Set $D_0 = D$ and $D_{i+1} = \sigma_i(D_i)$ for $0 \leq i < k$. D_i is an anti-canonical divisor of Z_i, i.e., $D_i \in |-\omega_{Z_i}|$. Since $C_i = \sigma_i^{-1}(z_i)$ is the exceptional curve of the first kind, we have $D_i \cdot C_i = 1$ and thus $z_i \in D_{i+1}$. Next set $H_0 = H$ and $H_{i+1} = \sigma_i(H_i)$ for $0 \leq i < k$. Obviously $C_i \neq H_i$ for every i. Assume $C_i \cap H_i = \phi$ for some i. We can assume moreover $C_j \cap H_j \neq \phi$ for $0 \leq j < i$. Then the strict inverse image $C_i' \subset Z$ of C_i in Z is an exceptional curve of the first kind and $C_i' \cap H = \phi$. However since Z is the minimal resolution, every exceptional curve of the first kind necessarily intersects with H. Thus one knows that $C_i \cap H_i \neq \phi$ for $0 \leq i < k$. We have:

Lemma 6.2. For $0 \leq i < k$, $z_i \in D_{i+1} \cap H_{i+1}$ and $D_0 \cap H_0 = \phi$.

Lemma 6.3. Assume $P = 3$. Then $Z = \bar{Z}$ and the branching locus B of $\pi: X \longrightarrow \mathbb{P}^2$ is a union of 6 lines passing through one point.

Proof. Assume $k \geq 1$. Let $F = p^{-1}p(z_{k-1}) \subset Z_k$. Note that $F \cdot H_k = 1$ since $p|_{H_k}$ is an isomorphism by Lemma 6.1. Thus $F' \cap H_{k-1} = \phi$ where F' is the strict inverse image of F by σ_{k-1}. However F' is an exceptional curve of the first kind and so is its strict inverse image F^* on Z. It contradicts to that Z is the minimal resolution since $F^* \cap H = \phi$. Therefore we have $k = 0$ and $Z = \bar{Z}$.

Set $F_t = p^{-1}(t)$ for $t \in G$. Note that $D \cdot F_t = 2$ by the adjunction formula for $F_t \cong \mathbb{P}^1$. Thus $\operatorname{Supp} D \cap F_t$ consists of one or two points for general $t \in G$. Assume that it is two points. Let D' be an irreducible component of D passing through one of these two points and D'' be an irreducible component of D passing through another point. The points $b_1 = \pi\rho(D')$ and $b_2 = \pi\rho(D'')$ are singular ones on B. Now since $F_t \cdot H = 1$, the morphism $\pi\rho$ maps F_t isomorphically onto a line in $\mathbb{P}^2$. Thus $b_1 \neq b_2$. However it implies that $\pi\rho(F_t)$ does not depend on t since it is a line passing through b_1 and b_2. We have a contradiction. Thus $\mathrm{Supp}\, D \cap F_t$ is one point for general $t \in G$. One sees that there is a section $s\colon G \longrightarrow Z$ such that $s(t) = \mathrm{Supp}\, D \cap F_t$ for general $t \in G$. Set $\bar{G} = s(G)$. We have $D = 2\bar{G}$ since $2\bar{G}$ is a component of D and since $(D-2\bar{G})\cdot H = 0$, $(D-2\bar{G})\cdot F_t = 0$. Set $x_0 = \rho(D) = \rho(\bar{G})$. The point $x_0 \in X$ is the unique singular point of X with $p_g(X, x) \geqq 1$.

Next we consider the line bundle $\mathcal{O}_Z(H-\bar{G})$. There is a line bundle M on G such that $\mathcal{O}_Z(H-\bar{G}) \cong p^*M$ because $(H-\bar{G})\, F_t = 0$ and thus $\mathcal{O}_Z(H-\bar{G})|_{F_t}$ is a trivial line bundle for every $t \in G$. We have $\deg M = (H-\bar{G})\cdot\bar{G} = 2$. Moreover note that $h^0(\mathcal{O}_Z(H-\bar{G})) \geqq 2$ since the divisor H defines a morphism $\pi\rho$ from Z to $\mathbb{P}^2$ and since $\pi\rho(\bar{G})$ is a point. By the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-\bar{G}) \longrightarrow \mathcal{O}_Z(H-\bar{G}) \longrightarrow \mathcal{O}_Z(H-\bar{G})|_H \longrightarrow 0$$

we have $h^0(M) = h^0(\mathcal{O}_Z(H-\bar{G})|_H) \geqq 2$. One sees that M is the dualizing sheaf ω_G of G by the Riemann-Roch theorem for curves. Let $\tau_1, \tau_2, \quad , \tau_6 \in G$ be the Weierstrass points on G. Setting $F_i = p^{-1}(\tau_i)$ we have $2F_i + \bar{G} \in |H|$ $(1 \leqq i \leqq 6)$.

Note that the last fact implies that $L_i = \pi\rho(F_i)$ is a component of the branching locus B. Since B is of degree 6, L_i is a line in $\mathbb{P}^2$ and $B = \bigcup_{i=1}^{6} L_i$. By definition L_i passes through $\pi(x_0)$ for every i. Q.E.D.

In what follows we assume that X has a singularity of type $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} and that $P = 2$. G is a smooth irreducible elliptic curve in this case.

Let $x_0 \in X$ be the point of type $\tilde{E}_8$, $T_{2,3,7}$ or E_{12}. We have another point $x_1 \in X$ with $p_g(X, x_1) = 1$. Let E be the connected component of the anti-canonical divisor D contained in $\rho^{-1}(x_0)$ and A be the connected component of D contained in $\rho^{-1}(x_1)$. We have $E^2 = -1$ and E is a smooth elliptic curve, a rational curve with one ordinary double point or a rational curve with one ordinary cusp according as x_0 is of type $\tilde{E}_8$, $T_{2,3,7}$ or E_{12}.

We set $E_0 = E$, $A_0 = A$, $E_{i+1} = \sigma_i(E_i)$ and $A_{i+1} = \sigma_i(A_i)$ for $0 \leqq i < k$.

Lemma 6.4. E_i and A_i are divisors on Z_i with $\operatorname{Supp} E_i \cap \operatorname{Supp} A_i = \phi$ for $0 \leqq i \leqq k$ and $E_i + A_i \in |-\omega_{Z_i}|$.

Proof. We use induction on i. The case $i = 0$ is trivial. Assume it holds for some i with $0 \leqq i < k$. Set $C_i = \sigma_i^{-1}(z_i)$.

Note that either (a) $C_i \cap \operatorname{Supp} E_i = \phi$ or (b) $C_i \cap \operatorname{Supp} A_i = \phi$ holds. Indeed assume both (a) and (b) do not hold. We deduce a contradiction. If $C_i \cdot A_i \leq 0$, then C_i is a component of A_i under this assumption and we have $\mathrm{Supp}\, A_i \cap \mathrm{Supp}\, E_i \supset C_i \cap \mathrm{Supp}\, E_i \neq \phi$, a contradiction. Thus $C_i \cdot A_i > 0$. Similarly we have $C_i \cdot E_i > 0$. On the other hand $1 = -C_i \cdot \omega_{Z_i} = C_i \cdot A_i + C_i \cdot E_i \geq 2$, which is a contradiction again. Thus either (a) or (b) holds. If (a) holds, then σ_i is an isomorphism on a neighbourhood of $\mathrm{Supp}\, E_i$ and thus E_{i+1} is a divisor with $\mathrm{Supp}\, E_{i+1} \cap \mathrm{Supp}\, A_{i+1} = \phi$. Then if A_{i+1} is not a divisor, $A_i = mC_i$ for some positive integer m. However we have $-1 = \omega_{Z_i} \cdot C_i = -mC_i^2 = m$, a contradiction. Thus A_{i+1} is also a divisor. Even under (b) we have tha same conclusion.

Moreover since $\sigma_{i*}\omega_{Z_i} = \omega_{Z_{i+1}}$, one has that $E_{i+1} + A_{i+1} \in |-\omega_{Z_{i+1}}|$. Q.E.D.

Lemma 6.5. For some section $s_i \colon G \longrightarrow \bar{Z}$ of p $(i = 1, 2)$, $A = s_1(G)$ and $E = s_2(G)$.

Proof. Let $F = p^{-1}(\tau)$ be the fibre over a general point $\tau \in G$. Note that $F \cdot (A_k + E_k) = -F\, \omega_{\bar{Z}} = 2$ holds since $0 = (F^2 + \omega_{\bar{Z}} \cdot F)/2 + 1$ and $F^2 = 0$. Assume $F\, A_k = 0$, then $A_k = mF_t$ for some positive integer m and for some $F_t = p^{-1}(t)$ with $t \in G$. We have $A_k\, E_k = mF_t \cdot E_k = 2m > 0$, which contradicts to Lemma 6.4. Thus $F\, A_k > 0$. Similarly we have $F \cdot E_k > 0$. One sees $F \cdot A_k = F \cdot E_k = 1$. Note that this equality implies that $A_k = s_1(G) + \sum_{j=1}^{q} F_{t_j}$ and $E_k = s_2(G) + \sum_{j=1}^{r} F_{t'_j}$ for some section $s_1, s_2 \colon G \longrightarrow \bar{Z}$ of p and for some points t_j, $t'_j \in G$. Since $s_i(G) \cdot F_t > 0$ for $i = 1, 2$, $t \in G$, one can conclude that $A_k = s_1(G)$ and $E_k = s_2(G)$ by Lemma 6.4.

Now since σ is a composition of blowing-ups of infinitely near singular points on $A_k \cup D_k$, A and E are also smooth elliptic curves. Q.E.D.

Corollary 6.6. Both $x_0 \in X$ and $x_1 \in X$ are simple elliptic singularities of multiplicity 2.

Lemma 6.7. There exists a birational morphism to a relatively minimal model $\sigma' : Z \longrightarrow \bar{Z}'$ such that $\sigma'(E)^2 = E^2 = -1$.

Proof. For a contraction $\sigma : Z \longrightarrow \bar{Z}$ to a relatively minimal model we set $\alpha(\bar{Z}) = \sigma(E)^2 - E^2$. It suffices to show that if $\alpha(\bar{Z}) > 0$ then we have another contraction $\sigma' : Z \longrightarrow \bar{Z}'$ to a relatively minimal model such that $\alpha(\bar{Z}') = \alpha(\bar{Z}) - 1$.

Assume $\alpha(\bar{Z}) > 0$. By exchanging the order of blowing-ups we may assume that the center $z_{k-1} \in Z_k$ of σ_{k-1} belongs to E_k. Set $F = p^{-1}(p(z_{k-1}))$. F is a smooth rational curve and the strict inverse image F' of F by σ_{k-1} is an exceptional curve of the first kind. Moreover $F' \cap E_{k-1} = \phi$ since E_k is a section of p. Let $\tau : Z_{k-1} \longrightarrow \bar{Z}'$ be the contraction of F'. Then obviously $\sigma' = \tau\sigma_{k-2} \quad \sigma_0 : Z \longrightarrow \bar{Z}'$ has the desired property. Q.E.D.

By Lemma 6.7, we can assume that $z_{i-1} \in A_i$ for $1 \leqq i \leqq k$ in (6.1).

In what follows we set this assumption. Then we have $k = \sigma(A)^2 - A^2 = 1-A^2$ since $\sigma(A)^2 = -\sigma(E)^2 = 1$.

Lemma 6.8. $A^2 = -1$.

Proof. Since A is an exceptional curve of the resolution $\rho: Z \longrightarrow X$, we have $A^2 \leqq -1$. If $A^2 \leqq -3$, then the contracted singular point x_1 is not a double point since A is a smooth elliptic curve. (Cf. Saito [18])

Assume $A^2 = -2$. Then $k = 3$ and $Z_3 = \bar{Z}$. Let m_i be the multiplicity of H_i at z_i. By Lemma 6.2, we have $m_i \geqq 1$ for $1 \leqq i \leqq 3$. On the other hand since $H_3 \cap E_3 = \phi$, H_3 is numerically equivalent to nA_3 for some integer n. Since $H \cdot A = 0$, we have $nA_3{}^2 - m_1 - m_2 - m_3 = 0$. Moreover $2 = n^2 - m_1{}^2 - m_2{}^2 - m_3{}^2$ since $H^2 = 2$. They imply that $m_1m_2 + m_2m_3 + m_3m_1 = 1$. However the left-hand-side is greater than or equal to 3 since $m_i \geqq 1$, which is a contradiction. Thus one sees $A^2 = -1$. Q.E.D.

Corollary 6.9. The point $x_1 = \rho(A)$ is also of type $\tilde{E}_8$.

Proposition 6.10. Assume that the branched double covering X over $\mathbb{P}^2$ branching along a reduced sextic curve has a singularity of type $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} and that $\sum_{x \in X} p_g(X, x) = 2$. Then the configuration of singurarities on X is either $2\tilde{E}_8$ or $2\tilde{E}_8 + A_1$.

Proof. First of all we note the following fact. Let $f: \mathbb{P}^1 \longrightarrow Z$ be an arbitrary morphism from $\mathbb{P}^1$. Then the composition $p\sigma f$ is a morphism from $\mathbb{P}^1$ to an elliptic curve. Thus its image is a point. Namely one sees that any rational curve in Z is either a strict inverse image of $F_t = p^{-1}(t)$ for some $t \in G$ or an exceptional curve of σ.

Note moreover that $k = 2$ since $A^2 = -1$.

If $\sigma_1(z_0) \neq z_1$, there is no smooth irreducible rational curve with the self-intersection number -2 on Z and thus the configuration is $2\tilde{E}_8$.

Assume $\sigma_1(z_0) = z_1$. Let $F_1 = p^{-1}(p(z_1))$ and F_1' be the strict inverse image of F_1 by σ_1. Since F_1 and A_2 intersect transversally at z_1, z_0 does not lie on F_1'. Thus $(F_1^{\bullet})^2 = -1$ where $F_1^{\bullet}$ is the strict inverse image of F_1' by σ_0. Next note that the strict inverse image C_1' of $C_1 = \sigma_1^{-1}(z_1)$ is a smooth irreducible rational curve with $C_1'^2 = -2$. We have of course $C_2^2 = -1$ for $C_0 = \sigma_0^{-1}(z_0)$. We see that the configuration for X is $2\tilde{E}_8$ or $2\tilde{E}_8 + A_1$. Q.E.D.

Lemma 6.11. There exists a reduced plane sextic curve whose configuration of singularities is $2\tilde{E}_8$. (resp. $2\tilde{E}_8+A_1$.)

Proof. The following figures give the examples.

Figure 6.1.

We now complete all the proof of our main theorems.

References

[1] Arnold, V.I.: Local normal forms of functions, Invent. Math. 35, p.87-109 (1976)

[2] Artin, M.: On isolated rational singularities of surfaces, Amer. J. Math. 88, p.129-137 (1966)

[3] Bourbaki, N.: Groupes et algèbre de Lie, Chap. IV, V, VI, Hermann, Paris (1968)

[4] Deligne, P.: Théorie de Hodge II, III, Publ. Math. I.H.E.S. 40, p.5-58 (1971), 44, p.5-77 (1972)

[5] Demazure, M.: Surfaces de Del Pezzo, II, III, IV et V, Lecture Notes in Math. No.77, Springer, Berlin Heidelberg New-York (1980)

[6] Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras, A.M.S. Translations (2) 6 p.111-244 (1957)

[7] Grauert, H.: Über Modifikationen und exzeptionalle analytische Memgen, Math. Ann. 146, p.331-368 (1962)

[8] Hartshorne, R.: Algebraic Geometry, Springer, Berlin Heidelberg New-York (1977)

[9] Kawamata, Y.: A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann. 261, p.43-46 (1982)

[10] Looijenga, E: Rational surfaces with an anti-canonical cycle, Annals of Math. 114, p.267-322 (1981)

[11] Laufer, H.B.: On rational singularities, Amer. J. Math. 94, p.597-608 (1972)

[12] Matsumura, H.: Commutative algebra, W. A. Benjamin Co., New-York (1970)

[13] Mérindol, J.-Y.: Les singurarités simple elliptiques, leur déformations, les surfaces de Del Pezzo et les transformations quadriques, Ann. scient. Éc. Norm. Sup. 4^e série t. 15, p.17-44 (1982)

[14] Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.E.S. 9, p.5-22 (1961)

[15] Naruki, I., Urabe, T.: On degenerate Del Pezzo surfaces of degree 1, 2, (preprint 1981)

[16] Pinkham, H.: Simple elliptic singularities, Del Pezzo surfaces and Cremona transformations, Proc. of Synposia in Pure Math., 30, p.69-70 (1977)

[17] Saint-Donat, B.: Projective models of K3 surfaces, Amer. J. Math. Vol. 96, No. 4, p.602-639 (1974)

[18] Saito, K.: Einfach-elliptische Singularitäten, Invent. Math. 23, p.289-325 (1974)

[19] Shafarevich, I. R.: Algebraic surfaces, Proc. Steklov Inst. Math. 75 (1965) (trans. by A.M.S. 1967)

[20] Shah, J.: Degenerations of K3 surfaces of degree 4, Trans. Amer. Math. Soc., Vol. 263, No. 2, p.271-308 (1981)

[21] Umezu, Y.: Quartic surfaces of elliptic ruled type (preprint 1982)

[22] Du Val, P.: On isolated singularities which do not affect the conditions of adjunction, III, Proc. Cambridge Philos. Soc., 30, p.483-491 (1934)

[23] Wagreich, P.: Elliptic singularities of surfaces, Amer. J. Math. 92, p.419-454 (1970)

[24] Zariski, O.: On the problem of existence of algebraic function of two variables possessing a given branch curve, Amer. J. Math. 51, p.305-328 (1929)

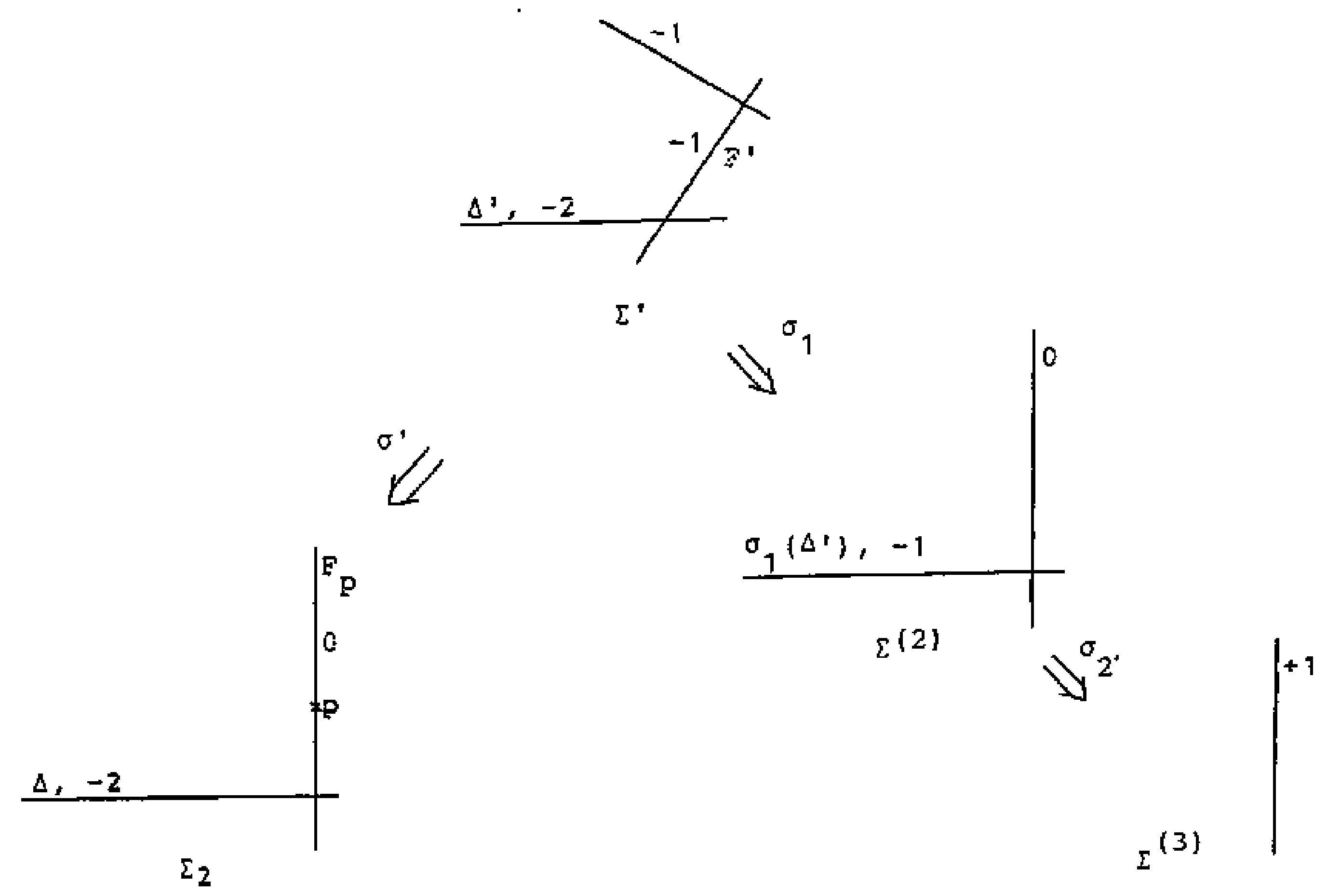


Figure 1.1.: T. Urabe, On quartic surfaces and sextic curves

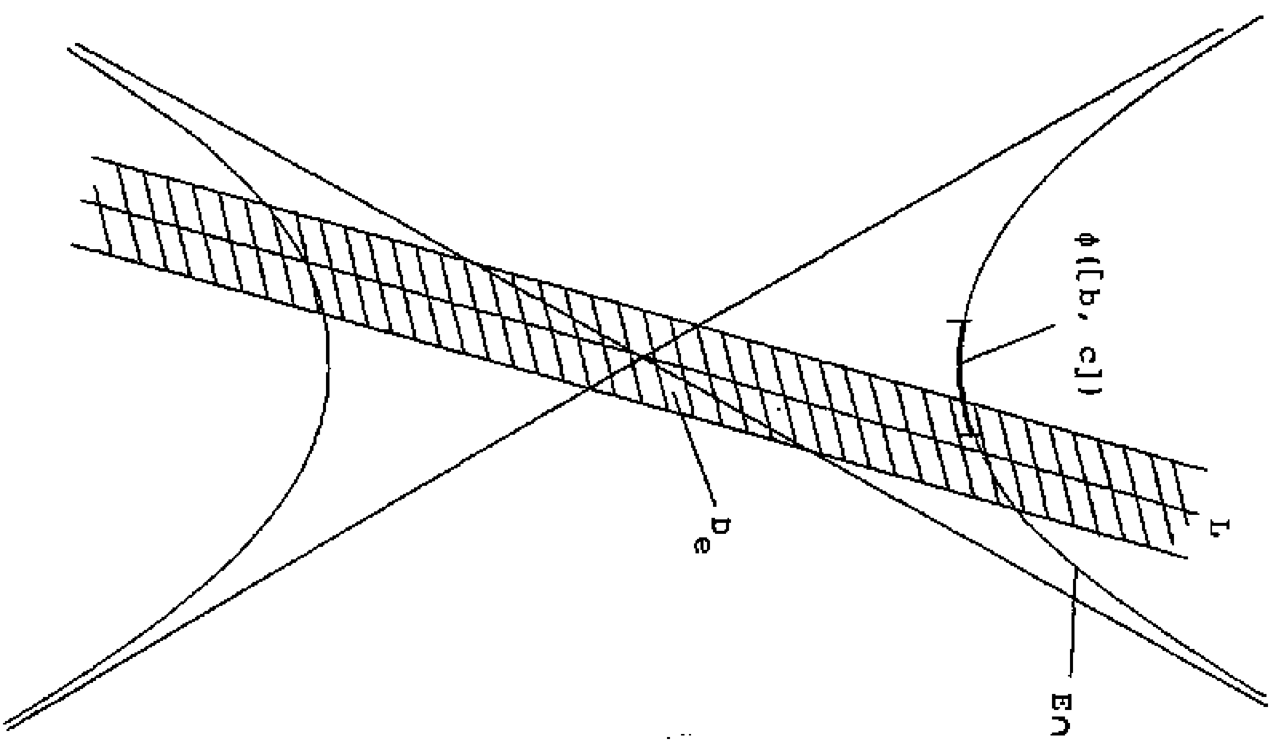


Figure 4.1.: T. Urabe,
On quartic surfaces and sextic curves

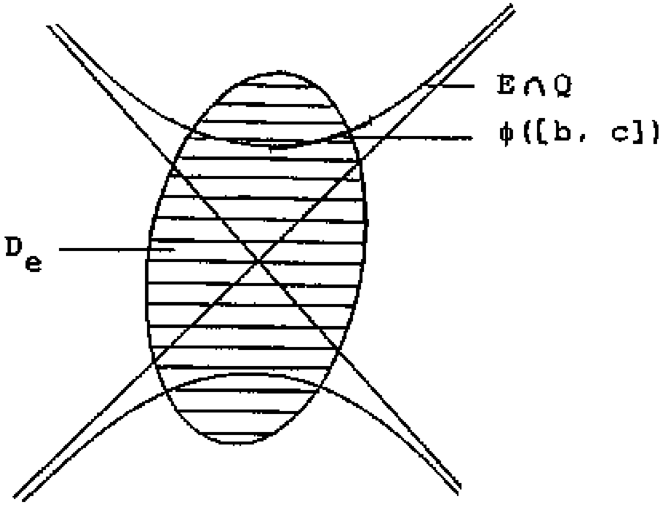


Figure 4.2.: T. Urabe,
On quartic surfaces and sextic curves

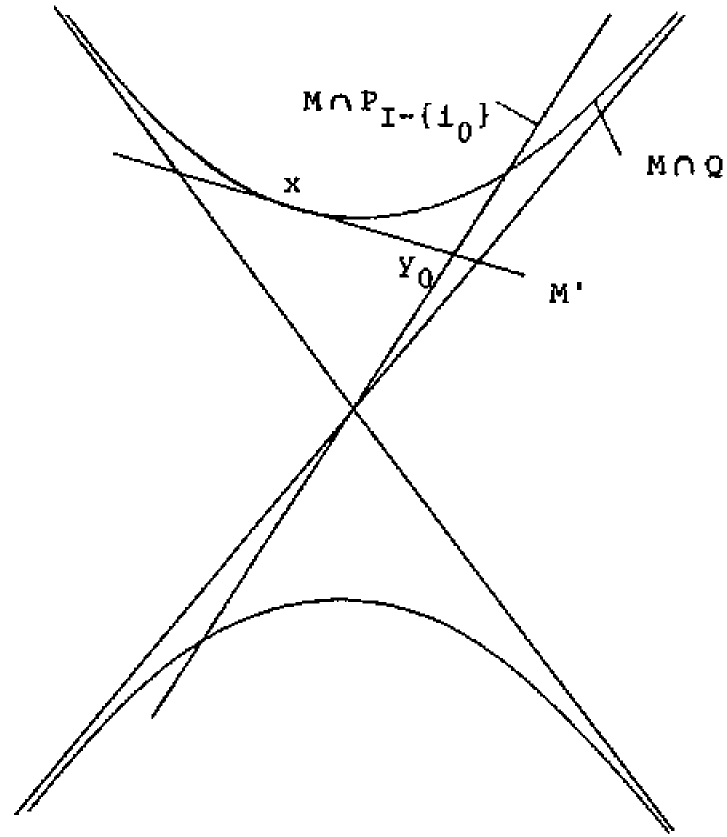


Figure 4.3.: T. Urabe,
On quartic surfaces and sextic curves

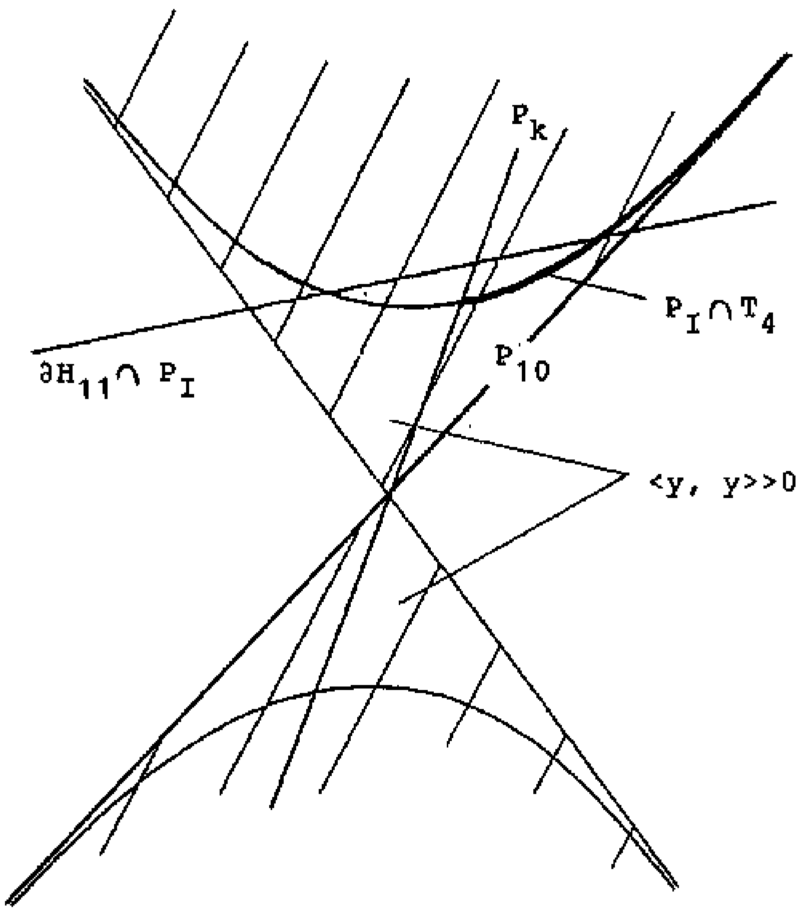


Figure 4.5.: T. Urabe,
On quartic surfaces and sextic cu...

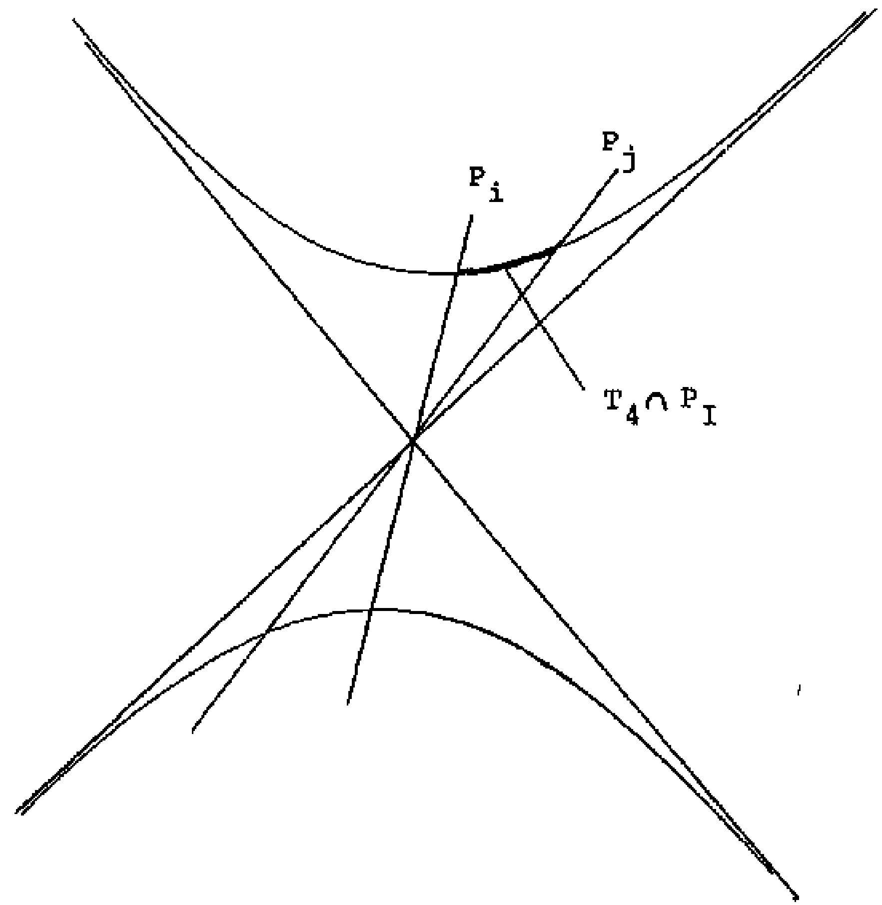


Figure 4.4.: T. Urabe,
On ...

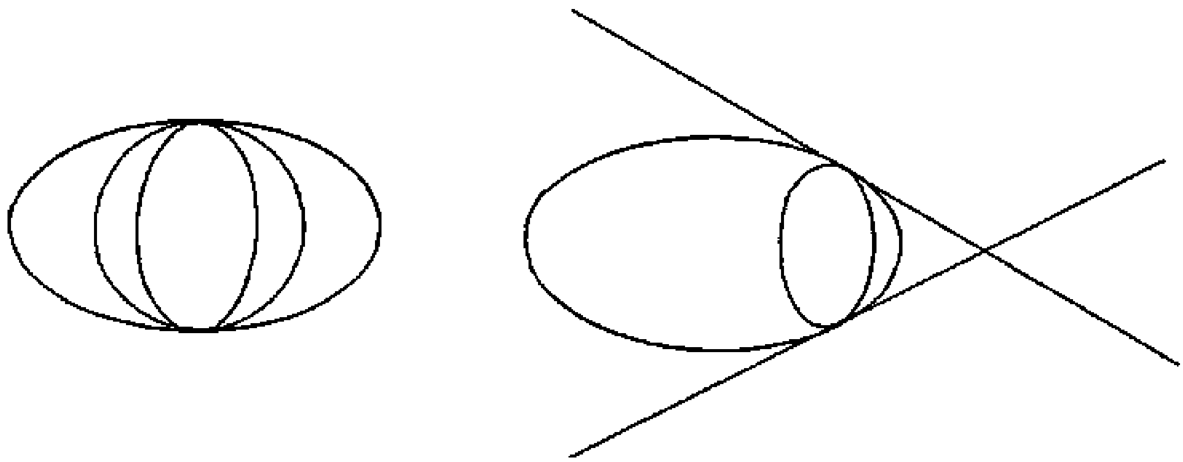


Figure 6 1 : T Urabe, On quartic surfaces and sextic curves